

Partner orbits and action differences on compact factors of the hyperbolic plane. Part II: Higher-order encounters

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Abstract

Physicists have argued that periodic orbit bunching leads to universal spectral fluctuations for chaotic quantum systems. To establish a more detailed mathematical understanding of this fact, it is first necessary to look more closely at the classical side of the problem and determine orbit pairs consisting of orbits which have similar actions. We specialize to the geodesic flow on compact factors of the hyperbolic plane as a classical chaotic system. The companion paper (Huynh and Kunze, 2015) proved the existence of a unique periodic partner orbit for a given periodic orbit with a small-angle self-crossing in configuration space that is a 2-encounter and derived an estimate for the action difference of the orbit pair. In this paper, we provide an inductive argument to deal with higher-order encounters: we prove that a given periodic orbit including an L -parallel encounter has $(L - 1)! - 1$ partner orbits; we construct partner orbits and give estimates for the action differences between orbit pairs.

Keywords: geodesic flow, periodic solution, partner orbit, higher-order encounter

1 Introduction

In the semi-classical limit chaotic quantum systems very often exhibit universal behavior, in the sense that several of their characteristic quantities agree with the respective quantities found for certain ensembles of random matrices. Via trace formulas, such quantities can be illustrated as suitable sums over the periodic orbits of the underlying classical dynamical system. For instance, the two-point correlator function is expressed by a double sum over periodic orbits

$$K(\tau) = \left\langle \frac{1}{T_H} \sum_{\gamma, \gamma'} A_\gamma A_{\gamma'}^* e^{\frac{i}{\hbar}(S_\gamma - S_{\gamma'})} \delta\left(\tau T_H - \frac{T_\gamma + T_{\gamma'}}{2}\right) \right\rangle, \quad (1.1)$$

where $\langle \cdot \rangle$ abbreviates the average over the energy and over a small time window, T_H denotes the Heisenberg time and A_γ , S_γ , and T_γ are the amplitude, the action, and the period of the orbit γ , respectively.

The diagonal approximation $\gamma = \gamma'$ to (1.1) studied by Hannay/Ozorio de Almeida [9] and Berry [3] in the 1980's contributes to the first order 2τ to (1.1); see also [14]. To next orders, as $\hbar \rightarrow 0$, the main term from (1.1) arises owing to those orbit pairs $\gamma \neq \gamma'$ for which the action difference $S_\gamma - S_{\gamma'}$ is 'small'. This was first considered by Sieber and Richter [20, 21], who predicted that a given periodic orbit with a small-angle self-crossing in configuration space will admit a partner orbit with almost the same action. The original orbit and its partner are called a Sieber-Richter pair. In phase space, a Sieber-Richter pair contains a region where two stretches of each orbit are almost mutually time-reversed and one addresses this region as a *2-encounter* or, more strictly, a *2-antiparallel encounter*; the '2' stands for two orbit stretches which are close in configuration space, and 'antiparallel' means that the two stretches have opposite directions (see Figure 1). It was shown in [21] that Sieber-Richter pairs contribute to the spectral form factor (1.1) the second order term $-2\tau^2$, and it turned out that the result agreed with what is obtained using random matrix theory [6], for certain symmetry classes.

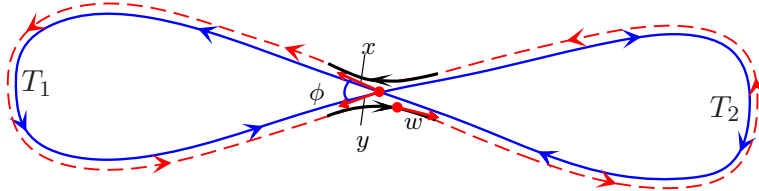


Figure 1: Example of a Sieber-Richter pair

This discovery prompted an increased research activity on the subject matter in the following years and finally led to an expansion

$$K(\tau) = 2\tau - \tau \ln(1 + 2\tau) = 2\tau - 2\tau^2 + 2\tau^3 + \dots$$

for the orthogonal ensemble (the symmetry class relevant for time-reversal invariant systems) to all orders in τ , by including the higher-order encounters also; see [10, 16, 17, 18], and in addition [8, 15], which provide much more background and many further references.

To establish a more detailed mathematical understanding, we start on the classical side and try to prove the existence of partner orbits and derive good estimates for the action differences of the orbit pairs. For 2-encounters this was done in the previous work [12], where we considered the geodesic flow on compact factors of the hyperbolic plane; in this case the action of a periodic orbit is proportional to its length. It was shown in [12] that a T -periodic orbit of the geodesic flow crossing itself in configuration space at a time T_1 has a unique partner orbit that remains $9|\sin(\phi/2)|$ -close to the original one and the action difference between them is approximately equal $\ln(1 - (1 + e^{-T_1})(1 + e^{-(T-T_1)})\sin^2(\phi/2))$ with the estimated error $12\sin^2(\phi/2)e^{-T}$, where ϕ is the crossing angle (see Figure 1), and this proved the accuracy of Sieber/Richter's prediction in [21] mentioned above. In

this paper, we continue considering this hyperbolic dynamical system to deal with the technically more involved higher-order encounters.

In the physics community this system is often called the Hadamard-Gutzwiller model, and it has frequently been studied [5, 10, 19]; further related work includes [8, 18, 22]. For instance, Heusler et al. [10] identified the term $2\tau^3$ and it was shown that there are 5 families of orbit pairs contributing to $2\tau^3$: 3 families of orbits with two single 2-encounters and 2 families of orbits with 3-encounters. In that way one obtains whole bunches of periodic orbits with controlled and small mutually action difference. Generalizing these ideas, in [17, 18] the expansion of $K(\tau)$ to all orders in τ could be derived. Here a key point was to consider encounters where more than two orbit stretches are involved; see also [8, 15, 22]. We speak of an *L-encounter* when L stretches of a periodic orbit are mutually close to each other up to time reversal. For a precise definition one may pick one of L encounter stretches as a reference and demand that none of the $L - 1$ companions be further away than some small distance; see [1]. In other words, all the L stretches must intersect a small Poincaré section. The orbit enters the *encounter region* through *entrance ports* and leaves it through *exit ports*. Using hyperbolicity, one can switch connections between entrance ports and exit ports to get new orbits which still remain close to the original one; and they are called *partner orbits*. However, not all the connections give genuine periodic orbits since some of them decompose to several shorter orbits (called pseudo-orbits; see [12]). Müller et al. [18] used combinatorics to count the number of partner orbits and provided an approximation for the action difference, but a construction of partner orbits and an error bound of the approximation have not been derived. Furthermore, it is necessary to arrive a mathematical definition for ‘encounters’, ‘partner orbits’, and to introduce the notions of ‘beginning’, ‘end’, and ‘duration’ of an encounter.

The paper is organized as follows. In Section 2, after giving some background material, we introduce another version of Poincaré sections and a respective version of the Anosov closing lemma. In the case that the space is compact, a Poincaré section with small radius can be identified with a square, so every point in a Poincaré section can be represented by a unique couple $(u, s) \in \mathbb{R}^2$ called *unstable* and *stable coordinates*. In addition, we provide a ‘connecting lemma’ to connect 2 orbits in a pseudo-orbit. In this way, one can construct a genuine periodic orbit close to a given pseudo-orbit. In Section 3, after providing mathematically rigorous definitions of ‘encounters’, ‘partner orbits’, ‘orbit pairs’, etc, we provide an inductive argument to construct partner orbits for a given orbit with a single L -parallel encounter. The first step of the inductive argument stated in Theorem 3.1 shows the existence of a unique partner orbit for a given orbit with a 3-parallel encounter. The main result of the current paper is Theorem 3.2 which proves that there exist $(L - 1)! - 1$ partner orbits for a given periodic orbit with an L -parallel encounter such that any two piercing points are not too close. This condition is expressed by differences of stable and unstable coordinates of the piercing points which guarantees that the encounter stretches are separated by non-vanishing loops and whence the partner orbits do not coincide. We use combinatorics to count the number of partner orbits and apply the connecting lemma, the Anosov closing lemmas to construct partner orbits. The action difference between the orbit pairs can be approximated by terms of coordinates of the piercing points with a precisely estimated error.

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2 Preliminaries

We consider the geodesic flow on compact Riemann surfaces of constant negative curvature. In fact this flow has had a great historical relevance for the development of the whole theory of hyperbolic dynamical systems or Anosov systems. It is well-known that any compact orientable surface with a metric of constant negative curvature is isometric to a factor $\Gamma \backslash \mathbb{H}^2$, where $\mathbb{H}^2 = \{z = x + iy \in \mathbb{C} : y > 0\}$ is the hyperbolic plane endowed with the hyperbolic metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$ and Γ is a discrete subgroup of the projective Lie group $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R}) / \{\pm E_2\}$; here $\mathrm{SL}(2, \mathbb{R})$ is the group of all real 2×2 matrices with unity determinant, and E_2 denotes the unit matrix. The group $\mathrm{PSL}(2, \mathbb{R})$ acts transitively on \mathbb{H}^2 by Möbius transformations $z \mapsto \frac{az+b}{cz+d}$. If the action is free (of fixed points), then the factor $\Gamma \backslash \mathbb{H}^2$ has a Riemann surface structure. Such a surface is a closed Riemann surface of genus at least 2 and has the hyperbolic plane \mathbb{H}^2 as the universal covering. The geodesic flow $(\varphi_t^{\mathcal{X}})_{t \in \mathbb{R}}$ on the unit tangent bundle $\mathcal{X} = T^1(\Gamma \backslash \mathbb{H}^2)$ goes along the unit speed geodesics on $\Gamma \backslash \mathbb{H}^2$. This means that every orbit under the geodesic flow $(\varphi_t^{\mathcal{X}})_{t \in \mathbb{R}}$ is a geodesic on \mathcal{X} which is the projection of a geodesic on \mathbb{H}^2 . In addition, every oriented unit speed closed geodesic on $\Gamma \backslash \mathbb{H}^2$ is a periodic orbit of the geodesic flow $(\varphi_t^{\mathcal{X}})_{t \in \mathbb{R}}$ on $\mathcal{X} = T^1(\Gamma \backslash \mathbb{H}^2)$.

On the other hand, the unit tangent bundle $T^1(\Gamma \backslash \mathbb{H}^2)$ is isometric to the quotient space $\Gamma \backslash \mathrm{PSL}(2, \mathbb{R}) = \{\Gamma g, g \in \mathrm{PSL}(2, \mathbb{R})\}$, which is the system of right co-sets of Γ in $\mathrm{PSL}(2, \mathbb{R})$, by an isometry Ξ . Then the geodesic flow $(\varphi_t^{\mathcal{X}})_{t \in \mathbb{R}}$ can be equivalently described as the natural ‘quotient flow’ $\varphi_t^{\mathcal{X}}(\Gamma g) = \Gamma g a_t$ on $X = \Gamma \backslash \mathrm{PSL}(2, \mathbb{R})$ associated to the flow $\phi_t(g) = g a_t$ on $\mathrm{PSL}(2, \mathbb{R})$ by the conjugate relation

$$\varphi_t^{\mathcal{X}} = \Xi^{-1} \circ \varphi_t^X \circ \Xi \quad \text{for all } t \in \mathbb{R}.$$

Here $a_t \in \mathrm{PSL}(2, \mathbb{R})$ denotes the equivalence class obtained from the matrix $A_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$. In fact, there are one-to-one correspondences between the collection of all periodic orbits of $(\varphi_t^{\mathcal{X}})_{t \in \mathbb{R}}$ (denoted by \mathcal{PO}_X), the collection of all oriented unit speed closed geodesics on Y (denoted by \mathcal{CG}_Y), and the conjugacy classes in Γ (denoted by \mathfrak{C}_Γ). The period T of a periodic orbit in \mathcal{PO}_X and the length l of the corresponding closed geodesic in \mathcal{CG}_Y are related by $T = l = 2 \operatorname{arccosh}(\frac{\operatorname{tr}(\gamma)}{2})$, where γ is a representative of the respective conjugacy class in \mathfrak{C}_Γ .

There are some more advantages to work on $X = \Gamma \backslash \mathrm{PSL}(2, \mathbb{R})$ rather than on $\mathcal{X} = T^1(\Gamma \backslash \mathbb{H}^2)$. One can calculate explicitly the stable and unstable manifolds at a point x to be

$$W_X^s(x) = \{\theta_t^X(x), t \in \mathbb{R}\} \quad \text{and} \quad W_X^u(x) = \{\eta_t^X(x), t \in \mathbb{R}\},$$

where $(\theta_t^X)_{t \in \mathbb{R}}$ and $(\eta_t^X)_{t \in \mathbb{R}}$ are the *horocycle flow* and *conjugate horocycle flow* defined by $\theta_t^X(\Gamma g) = \Gamma g b_t$ and $\eta_t^X(\Gamma g) = \Gamma g c_t$; here $b_t, c_t \in \mathrm{PSL}(2, \mathbb{R})$ denote the equivalence classes obtained from

$B_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $C_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \in \text{SL}(2, \mathbb{R})$. The flow $(\varphi_t^X)_{t \in \mathbb{R}}$ is hyperbolic, that is, for every $x \in X$ there exists an orthogonal and $(\varphi_t^X)_{t \in \mathbb{R}}$ -stable splitting of the tangent space $T_x X$

$$T_x X = E^0(x) \oplus E^s(x) \oplus E^u(x)$$

such that the differential of the flow $(\varphi_t^X)_{t \in \mathbb{R}}$ is uniformly expanding on $E^u(x)$, uniformly contracting on $E^s(x)$ and isometric on $E^0(x) = \langle \frac{d}{dt} \varphi_t^X(x) |_{t=0} \rangle$. One can choose

$$E^s(x) = \text{span} \left\{ \frac{d}{dt} \theta_t^X(x) \Big|_{t=0} \right\} \quad \text{and} \quad E^u(x) = \text{span} \left\{ \frac{d}{dt} \eta_t^X(x) \Big|_{t=0} \right\}.$$

General references for this section are [2, 7, 13], and these works may be consulted for the proofs to all results which are stated above. In what follows, we will drop the superscript X from $(\varphi_t^X)_{t \in \mathbb{R}}$ to simplify notation.

2.1 Poincaré sections, stable and unstable coordinates

Recall that the Riemann surface $\Gamma \backslash \mathbb{H}^2$ is compact if and only if the quotient space $X = \Gamma \backslash \text{PSL}(2, \mathbb{R})$ is compact. Then there is $\sigma_0 > 0$ such that $d_G(g, \gamma g) \geq \sigma_0$ holds for all $g \in G = \text{PSL}(2, \mathbb{R})$ and $\gamma \in \Gamma \setminus \{e\}$. In the whole paper, we assume that X is compact. First we recall the definition of Poincaré sections in Part I [12].

Definition 2.1. *Let $x \in X$ and $\varepsilon > 0$. The Poincaré section of radius ε at x is*

$$\mathcal{P}_\varepsilon(x) = \{\Gamma(gc_u b_s) : |u| < \varepsilon, |s| < \varepsilon\},$$

where $g \in G$ is such that $x = \Gamma g$.

Lemma 2.1. *Let X be compact and $\varepsilon \in]0, \sigma_0/4[$. For every $y \in \mathcal{P}_\varepsilon(x)$ there exists a unique couple $(u, s) \in]-\varepsilon, \varepsilon[\times]-\varepsilon, \varepsilon[$ such that*

$$y = \Gamma(gc_u b_s)$$

for any $g \in G$ satisfying $x = \Gamma g$.

Proof. By definition such a couple (u, s) does exist. To show its uniqueness, suppose that $x = \Gamma g_1 = \Gamma g_2$ and $y = \Gamma g_1 b_{s_1} c_{u_1} = \Gamma g_2 b_{s_2} c_{u_2}$ for $g_1, g_2 \in G$ and $(u_1, s_1), (u_2, s_2) \in]-\varepsilon, \varepsilon[\times]-\varepsilon, \varepsilon[$. Then there are $\gamma, \gamma' \in \Gamma$ such that

$$\gamma g_1 = g_2 \quad \text{and} \quad \gamma' g_1 c_{u_1} b_{s_1} = g_2 c_{u_2} b_{s_2}.$$

Therefore

$$\begin{aligned} d_G(\gamma^{-1} \gamma' g_1 c_{u_1}, g_1 c_{u_1}) &= d_G(\gamma^{-1} g_2 c_{u_2} b_{s_2-s_1}, g_1 c_{u_1}) = d_G(g_1 c_{u_2} b_{s_2-s_1}, g_1 c_{u_1}) \\ &= d_G(b_{s_2-s_1}, c_{u_1-u_2}) \leq d_G(b_{s_1-s_2}, e) + d_G(c_{u_2-u_1}, e) \\ &\leq |s_1 - s_2| + |u_1 - u_2| < 2\varepsilon + 2\varepsilon < \sigma_0. \end{aligned}$$

From the property of σ_0 , it implies that $\gamma^{-1}\gamma' = e$, so that $\gamma = \gamma'$. Then $g_2c_{u_2}b_{s_2} = \gamma g_1c_{u_1}b_{s_1} = g_2c_{u_1}b_{s_1}$ yields $c_{u_2-u_1} = b_{s_1-s_2}$, and consequently $s_1 = s_2, u_1 = u_2$ by considering matrices. \square

Thus for $\varepsilon \in]0, \sigma_0/4[$ the mapping

$$\mathcal{P}_\varepsilon(x) \rightarrow]-\varepsilon, \varepsilon[\times]-\varepsilon, \varepsilon[, y \mapsto (u, s),$$

such that $y = \Gamma(gc_ub_s)$ defines a bijection.

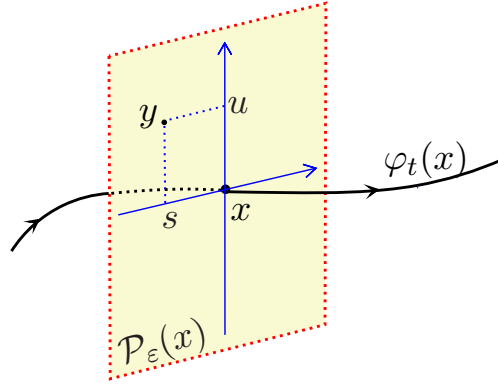


Figure 2: Coordinatization of Poincaré section

Definition 2.2. The numbers $u = u(y)$ and $s = s(y)$ are called the unstable coordinate and the stable coordinate of y , respectively, and we write $y = (u, s)_x$ (see Figure 2).

We can also define Poincaré sections as the following.

Definition 2.3. Let $x \in X$ and $\varepsilon > 0$. The Poincaré section of radius ε at x is

$$\mathcal{P}'_\varepsilon(x) = \{\Gamma(gb_sc_u) : |s| < \varepsilon, |u| < \varepsilon\}$$

where $g \in G$ is such that $x = \Gamma g$.

Note that b_s and c_u are reversed as compared to $\mathcal{P}_\varepsilon(x)$. We also have the following result.

Lemma 2.2. Let X be compact and $\varepsilon \in]0, \sigma_0/4[$. For every $y \in \mathcal{P}'_\varepsilon(x)$ there exists a unique couple $(s, u) \in]-\varepsilon, \varepsilon[\times]-\varepsilon, \varepsilon[$ such that

$$y = \Gamma(gb_sc_u)$$

for any $g \in G$ satisfying $x = \Gamma g$.

Definition 2.4. Again the numbers $s = s(y)$ and $u = u(y)$ are called the stable coordinate and the unstable coordinate of y , respectively, and we write $y = (s, u)'_x$.

The following result shows that if two trajectories of the flow $(\varphi_t)_{t \in \mathbb{R}}$ intersect a Poincaré section and are sufficiently close in the past time (the future time, respectively), then the stable coordinates (unstable coordinates, respectively) of the piercing points are nearly equal. A similar approximation was used in [18] but left proven.

Theorem 2.1. *For every $\rho > 0$, there exists $\varepsilon = \varepsilon(\rho) > 0$ satisfying the following property. If $\delta \in]0, \frac{\sigma_0}{8}[$ and $x, x_i \in X$ are such that $x_i \in \mathcal{P}_\delta(x)$ with $x_i = (u_i, s_i)_x$, for $i = 1, 2$, then for any $T > 0$,*

- (a) *if $d_X(\varphi_t(x_1), \varphi_t(x_2)) < \varepsilon$ for $t \in [0, T]$ then $|u_1 - u_2| < \rho e^{-T}$;*
- (b) *if $d_X(\varphi_t(x_1), \varphi_t(x_2)) < \varepsilon$ for $t \in [-T, 0]$ then $|s_1 - s_2 - s_1 s_2(u_1 - u_2)| < \rho e^{-T}$.*
- (c) *if $d_X(\varphi_t(x_1), \varphi_t(x_2)) < \varepsilon$ for $t \in [-T, T]$ then $|u_1 - u_2| < \rho e^{-T}$ and $|s_1 - s_2| < \frac{3}{2}\rho e^{-T}$.*

Proof. Let $\rho > 0$ be given. According to Lemma 2.4 below, we can select $\varepsilon_1 = \varepsilon_1(\rho) > 0$ so that $d_G(u, e) < \varepsilon_1$ and $u = \pi(U) \in \mathbf{G}$ for $U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ yield $|u_{12}| + |u_{21}| < \rho$. Let $\varepsilon = \min \left\{ \frac{\sigma_0}{2+\sqrt{2}}, \varepsilon_1 \right\}$.

(a) & (b) Write $x = \Gamma g$ and $x_i = \Gamma g c_{u_i} b_{s_i}$ for $g \in \mathbf{G}$. Denote $c_i(t) = g c_{u_i} b_{s_i} a_t \in \mathbf{G}$. By the definition of d_X , for every $t \in [-T, T]$ there is $\gamma(t) \in \Gamma$ so that $d_X(\varphi_t(x_1), \varphi_t(x_2)) = d_G(c_1(t), \gamma(t)c_2(t)) < \varepsilon$. Using the property of σ_0 , we can show that $\gamma(t) = \gamma(0)$ for all $t \in [-T, T]$. In addition, since $d_G(g c_{u_1} b_{s_1}, g c_{u_2} b_{s_2}) = d_G(c_{u_1} b_{s_1}, c_{u_2} b_{s_2}) \leq |u_1| + |u_2| + |s_1| + |s_2| < 4\delta < \frac{\sigma_0}{2}$, we have

$$d_X(\varphi_0(x_1), \varphi_0(x_2)) = d_X(x_1, x_2) = d_G(c_{u_1} b_{s_1}, c_{u_2} b_{s_2}) = d_G(c_1(0), c_2(0)).$$

This means that we can take $\gamma(0) = e$ and hence $\gamma(t) = e$ for all $t \in [-T, T]$. Then $d_G(c_1(t), c_2(t)) < \varepsilon$ for all $t \in [-T, T]$, i.e.,

$$d_G(c_{u_1} b_{s_1} a_t, c_{u_2} b_{s_2} a_t) < \varepsilon \leq \varepsilon_1 \quad \text{for all } t \in [-T, T],$$

or equivalently,

$$d_G(a_{-t} b_{-s_2} c_{u_1 - u_2} b_{s_1} a_t, e) < \varepsilon_1 \quad \text{for all } t \in [-T, T].$$

Write $a_{-t} b_{-s_2} c_{u_1 - u_2} b_{s_1} a_t = [H(t)]$ with

$$H(t) = \begin{pmatrix} 1 - s_2(u_1 - u_2) & e^{-t}(s_1 - s_2 - s_1 s_2(u_1 - u_2)) \\ e^t(u_1 - u_2) & 1 + s_1(u_1 - u_2) \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

Whence the definition of ε_1 leads to

$$e^t |u_1 - u_2| + e^{-t} |s_1 - s_2 - s_1 s_2(u_1 - u_2)| < \rho \quad \text{for } t \in [-T, T].$$

In particular, $t = T$ and $t = -T$ imply (a) and (b), respectively.

(c) This follows from (a) and (b). □

We also have a reverse statement.

Theorem 2.2. For $\varepsilon \in]0, \frac{\sigma_0}{4}[$. Assume that $x, x_1, x_2 \in X$ are such that $x_i \in \mathcal{P}_{\frac{\varepsilon}{5}}(x)$ and $x_i = (u_i, s_i)_x$ for $i = 1, 2$. Then for any $T > 0$,

- (a) if $|u_1 - u_2| < \frac{\varepsilon}{2}e^{-T}$ then $d_X(\varphi_t(x_1), \varphi_t(x_2)) < \varepsilon$ for all $t \in [0, T]$;
- (b) if $|s_1 - s_2 - s_1 s_2(u_1 - u_2)| < \frac{\varepsilon}{2}e^{-T}$ then $d_X(\varphi_t(x_1), \varphi_t(x_2)) < \varepsilon$ for all $t \in [-T, 0]$;
- (c) if $|u_1 - u_2| + |s_1 - s_2| < \frac{\varepsilon}{2}e^{-T}$ then $d_X(\varphi_t(x_1), \varphi_t(x_2)) < \varepsilon$ for all $t \in [-T, T]$.

Proof. (a) Let $T > 0$ be given and fix $t \in [0, T]$. We have

$$\begin{aligned}
d_X(\varphi_t(x_1), \varphi_t(x_2)) &= d_X(\Gamma g c_{u_1} b_{s_1} a_t, \Gamma g c_{u_2} b_{s_2} a_t) \leq d_G(g c_{u_1} b_{s_1} a_t, g c_{u_2} b_{s_2} a_t) \\
&= d_G(c_{(u_1 - u_2)e^t} b_{s_1 e^{-t}}, b_{s_2 e^{-t}}) \leq d_G(c_{(u_1 - u_2)e^t} b_{s_1 e^{-t}}, e) + d_G(b_{s_2 e^{-t}}, e) \\
&\leq d_G(c_{(u_1 - u_2)e^t}, e) + d_G(b_{s_1 e^{-t}}, e) + d_G(b_{s_2 e^{-t}}, e) \leq |u_1 - u_2|e^t + (|s_1| + |s_2|)e^{-t} \\
&< \frac{\varepsilon}{2}e^{t-T} + 2\frac{\varepsilon}{5}e^{-t} < \varepsilon.
\end{aligned}$$

(b) First, we write $c_{u_i} b_{s_i} = b_{\tilde{s}_i} c_{\tilde{u}_i} a_{\tilde{\tau}_i}$ with

$$\tilde{\tau}_i = -2 \ln(1 + u_i s_i), \quad \tilde{u}_i = u_i(1 + u_i s_i), \quad \tilde{s}_i = \frac{s_i}{1 + u_i s_i}.$$

For $t \in [-T, 0]$, analogously to (a), we obtain

$$\begin{aligned}
d_X(\varphi_t(x_1), \varphi_t(x_2)) &\leq d_G(c_{u_1} b_{s_1} a_t, c_{u_2} b_{s_2} a_t) = d_G(b_{\tilde{s}_1} c_{\tilde{u}_1} a_{\tilde{\tau}_1} a_t, b_{\tilde{s}_2} c_{\tilde{u}_2} a_{\tilde{\tau}_2} a_t) \\
&= d_G(b_{(\tilde{s}_1 - \tilde{s}_2)e^{-t} c_{\tilde{u}_1} e^t a_{\tilde{\tau}_1}}, c_{\tilde{u}_2} e^t a_{\tilde{\tau}_2}) \\
&\leq d_G(b_{(\tilde{s}_1 - \tilde{s}_2)e^{-t}}, e) + d_G(c_{\tilde{u}_1} e^t, e) + d_G(c_{\tilde{u}_2} e^t, e) + d_G(a_{\tilde{\tau}_1}, e) + d_G(a_{\tilde{\tau}_2}, e) \\
&\leq |\tilde{s}_1 - \tilde{s}_2|e^{-t} + (|\tilde{u}_1| + |\tilde{u}_2|)e^t + \frac{1}{\sqrt{2}}(|\tilde{\tau}_1| + |\tilde{\tau}_2|) \\
&\leq |s_1 - s_2 - s_1 s_2(u_1 - u_2)|e^{-t} + (|\tilde{u}_1| + |\tilde{u}_2|)e^t + \frac{4}{\sqrt{2}}(|u_1 s_1| + |u_2 s_2|) \\
&< \frac{\varepsilon}{2}e^{-T-t} + \frac{2\varepsilon}{5} + \frac{4}{\sqrt{2}} \cdot \frac{2\varepsilon^2}{25} < \varepsilon.
\end{aligned}$$

(c) It follows from $|u_1 - u_2| + |s_1 - s_2| < \frac{\varepsilon}{2}e^{-T}$ that $|u_1 - u_2| < \frac{\varepsilon}{2}e^{-T}$ and $|s_1 - s_2 - s_1 s_2(u_1 - u_2)| < \frac{\varepsilon}{2}e^{-T}$ which prove (c) by (a) and (b). \square

2.2 Shadowing lemma, Anosov closing lemmas, and connecting lemma

We recall the shadowing lemma and reformulate the Anosov closing lemma in Part I [12];

Denote by $W_{X, \varepsilon}^s(x) = \{\Gamma(gb_t) : |t| < \varepsilon\}$ and $W_{X, \varepsilon}^u(x) = \{\Gamma(gc_t) : |t| < \varepsilon\}$ for $x = \Gamma g$ the local stable and local unstable manifold of x of size ε , respectively.

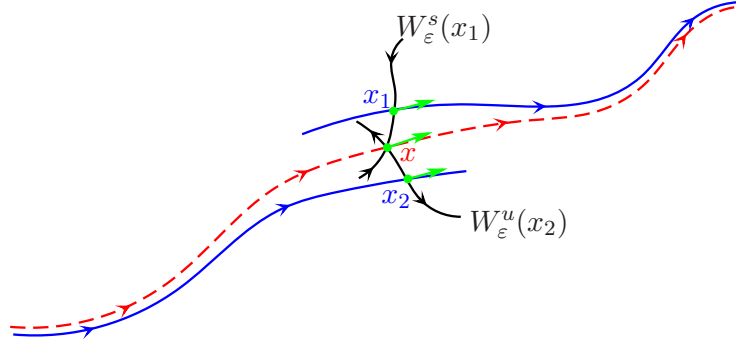


Figure 3: Shadowing lemma

Theorem 2.3 (Shadowing lemma). *If $\varepsilon > 0$, $x_1, x_2 \in X$, and $x \in W_{X,\varepsilon}^s(x_1) \cap W_{X,\varepsilon}^u(x_2)$, then*

$$d_X(\varphi_t(x_1), \varphi_t(x)) < \varepsilon e^{-t} \quad \text{for all } t \in [0, \infty[$$

and

$$d_X(\varphi_t(x_2), \varphi_t(x)) < \varepsilon e^t \quad \text{for all } t \in]-\infty, 0].$$

Theorem 2.4 (Anosov closing lemma I). *Suppose that $\varepsilon \in]0, \min\{\frac{1}{4}, \frac{\sigma_0}{8}\}[$, $x \in X$, $T \geq 1$, and $\varphi_T(x) \in \mathcal{P}_\varepsilon(x)$. Let $\varphi_T(x) = (u, s)_x$, in the notation from Definition 2.2. Then there are $x' \in \mathcal{P}_{2\varepsilon}(x)$ with $x' = (\sigma, \eta)_x$ and $T' \in \mathbb{R}$ so that*

$$\varphi_{T'}(x') = x' \quad \text{and} \quad d_X(\varphi_t(x), \varphi_t(x')) < 2|u| + |\eta| < 4\varepsilon \quad \text{for all } t \in [0, T].$$

Furthermore,

$$\left| \frac{T' - T}{2} - \ln(1 + us) \right| < 5|us|e^{-T},$$

$$e^{T'/2} + e^{-T'/2} = e^{T/2} + e^{-T/2} + use^{T/2},$$

and

$$|\sigma| < 2|u|e^{-T}, \quad |\eta - s| < 2s^2|u| + 2|s|e^{-T}.$$

Using the version of Poincaré sections in Definition 2.3, we have a respective statement for the Anosov closing lemma which will be also useful afterwards.

Theorem 2.5 (Anosov closing lemma II). *Suppose that $\varepsilon \in]0, \min\{\frac{1}{4}, \frac{\sigma_0}{8}\}[$, $x \in X$, $T \geq 1$, and $\varphi_T(x) \in \mathcal{P}'_\varepsilon(x)$. Let $\varphi_T(x) = (s, u)'_x$, in the notation from Definition 2.4. Then there are $x' \in \mathcal{P}'_{2\varepsilon}(x)$ with $x' = (\eta, \sigma)'_x$ and $T' \in \mathbb{R}$ so that*

$$\varphi_{T'}(x) = x \quad \text{and} \quad d_X(\varphi_t(x), \varphi_t(x')) \leq 2|u| + |\eta| < 4\varepsilon \quad \text{for all } t \in [0, T]. \quad (2.1)$$

Furthermore,

$$\left| \frac{T'}{2} - \frac{T}{2} \right| < 4|us|e^{-T}, \quad (2.2)$$

$$e^{T'/2} + e^{-T'/2} = e^{T/2} + e^{-T/2} + use^{-T/2}, \quad (2.3)$$

and

$$|\eta - s| \leq 2|s|e^{-T}, \quad |\sigma| < 2|u|e^{-T}. \quad (2.4)$$

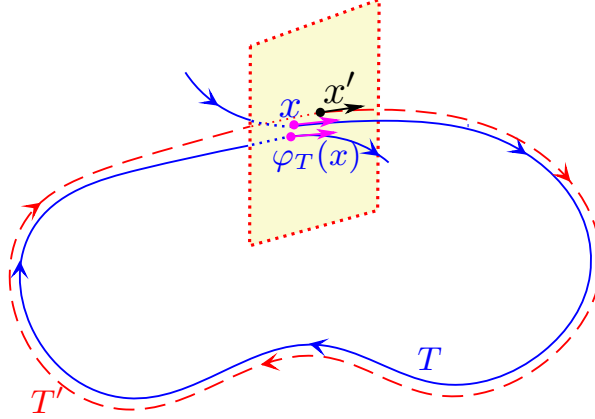


Figure 4: Anosov closing lemma

Proof. See Figure 4 for an illustration. By assumption, there are $s, u \in]-\varepsilon, \varepsilon[$ and $g \in \mathbf{G}$ such that $\Gamma g = x$ and $\varphi_T(x) = \Gamma g b_s c_u$. Then there is $\zeta \in \Gamma$ such that $\zeta g a_T = g b_s c_u$ or $\zeta = g b_s c_u a_{-T} g^{-1}$. The equation

$$ue^{-T}\eta^2 - ((1+su)e^{-T} - 1)\eta - s = 0$$

has a solution $\eta \in \mathbb{R}$ satisfying $|\eta - s| < 2|s|e^{-T}$ as well as $|\eta| < 2|s|$. Then

$$\sigma := \frac{u}{1 + (s - \eta)u - \eta u - e^T}$$

is well-defined and $|\sigma| < 2|u|e^{-T}$. Put $g' = g b_\eta c_\sigma \in \mathbf{G}$ and $x' = \Gamma g'$ to obtain $x' \in \mathcal{P}'_{2\varepsilon}(x)$. Defining

$$T' = T - 2 \ln(1 + (s - \eta)u), \quad (2.5)$$

we have

$$\left| \frac{T'}{2} - \frac{T}{2} \right| = |\ln(1 + (s - \eta)u)| \leq 2|s - \eta||u| < 4|us|e^{-T},$$

which is (2.2). Similarly to the proof of the Anosov closing lemma I, we can check that $\zeta g b_\eta c_\sigma a_{T'} = g b_\eta c_\sigma$ and hence $\varphi_{T'}(x') = x'$ and we also have the latter of (2.1). Due to $\zeta = g b_s c_u a_{-T} g^{-1} = g b_\eta c_\sigma a_{-T'} c_{-\sigma} b_{-\eta} g^{-1}$, this implies (2.3); and (2.4) can be done analogously to the Anosov closing lemma I. \square

Lemma 2.3 (Connecting lemma). *Let $x_j \in X$ be a T_j -periodic point of the flow $(\varphi_t)_{t \in \mathbb{R}}$, for $j = 1, 2$ and $T_1 + T_2 \geq 1$ and let $\varepsilon > 0$. If $x_2 \in \mathcal{P}_\varepsilon(x_1)$, then there is a periodic orbit 5ε -close to the orbits of x_1 and x_2 . More precisely, if $x_1 = \Gamma g_1$ and $x_2 = \Gamma g_1 c_u b_s$, then there are $x \in X$ and $T > 0$ such that $\varphi_T(x) = x$,*

$$\begin{aligned} d_X(\varphi_t(x), \varphi_t(x_1)) &< 5\varepsilon \quad \text{for all } t \in [0, T_1], \\ d_X(\varphi_{t+T_1}(x), \varphi_t(x_2)) &< 5\varepsilon \quad \text{for all } t \in [0, T_2], \end{aligned}$$

and

$$\left| \frac{T - (T_1 + T_2)}{2} - \ln(1 + us) \right| < 3|us|(e^{-T_1} + e^{-T_2}) + 8|us|e^{-T_1 - T_2}. \quad (2.6)$$

Furthermore, $x = \Gamma g_1 c_{ue^{-T_1 + \sigma}} b_\eta$, where $\sigma, \eta \in \mathbb{R}$ satisfy

$$|\eta - s| < 2s^2|u| + 2|s|e^{-T_1 - T_2} \quad \text{and} \quad |\sigma| < 2|u|e^{-T_1 - T_2}. \quad (2.7)$$

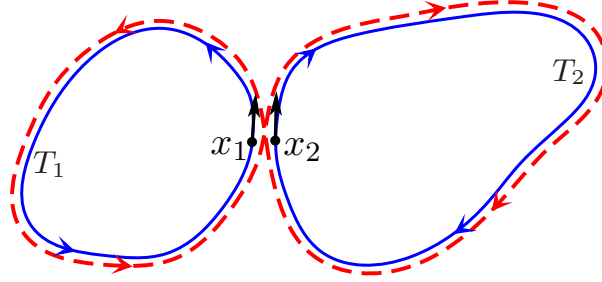


Figure 5: Reconnection a pseudo-orbit yields a genuine periodic orbit

Proof. See Figure 5 for an illustration. Write $x_j = \Gamma g_j$ with $g_j \in G$. Then $\Gamma g_2 = \Gamma g_1 c_u b_s$ and hence $\Gamma g_2 b_{-s} = \Gamma g_1 c_u =: w \in W_\varepsilon^s(x_2) \cap W_\varepsilon^u(x_1)$. By the shadowing lemma (Theorem 2.3),

$$d_X(\varphi_t(w), \varphi_t(x_2)) < \varepsilon e^{-t}, \quad \text{for all } t \geq 0 \quad (2.8)$$

and

$$d_X(\varphi_t(w), \varphi_t(x_1)) < \varepsilon e^t, \quad \text{for all } t \leq 0. \quad (2.9)$$

For $\hat{w} = \varphi_{-T_1}(w)$, we verify that $\varphi_{T_1+T_2}(\hat{w}) \in \mathcal{P}_{2\varepsilon}(\hat{w})$. Indeed,

$$\begin{aligned} \varphi_{T_1+T_2}(\hat{w}) &= \Gamma g_2 b_{-s} a_{T_2} = \Gamma g_2 b_{-s} e^{-T_2} = \Gamma g_1 c_u b_{s(1-e^{-T_2})} = \Gamma g_1 c_u c_{-u} a_{-T_1} c_u b_{s(1-e^{-T_2})} \\ &= \Gamma g_1 c_u a_{-T_1} c_{u(1-e^{-T_1})} b_{s(1-e^{-T_2})} = \Gamma(g_1 c_u a_{-T_1}) c_{\tilde{u}} b_{\tilde{s}} \in \mathcal{P}_\varepsilon(\hat{w}), \end{aligned}$$

where

$$\tilde{u} = u(1 - e^{-T_1}) \quad \text{and} \quad \tilde{s} = s(1 - e^{-T_2}). \quad (2.10)$$

By the assumption $T_1 + T_2 \geq 1$, we apply the Anosov closing lemma I to get $x = \Gamma(g_1 c_u a_{-T_1}) c_\sigma b_\eta \in \mathcal{P}_{2\varepsilon}(\hat{w})$ and $T \in \mathbb{R}$ such that $\varphi_T(x) = x$,

$$d_X(\varphi_t(x), \varphi_t(\hat{w})) < 4\varepsilon \quad \text{for all } t \in [0, T_1 + T_2] \quad (2.11)$$

and

$$\left| \frac{T - (T_1 + T_2)}{2} - \ln(1 + \tilde{u}\tilde{s}) \right| \leq 5|\tilde{u}\tilde{s}|e^{-T_1-T_2}. \quad (2.12)$$

For $t \in [0, T_1]$, by (2.11) and (2.9)

$$\begin{aligned} d_X(\varphi_t(x), \varphi_t(x_1)) &\leq d_X(\varphi_t(x), \varphi_t(\hat{w})) + d_X(\varphi_t(\hat{w}), \varphi_t(x_1)) \\ &= d_X(\varphi_t(x), \varphi_t(\hat{w})) + d_X(\varphi_{t-T_1}(w), \varphi_{t-T_1}(x_1)) < 5\varepsilon. \end{aligned}$$

For $t \in [0, T_2]$, by (2.11) and (2.8)

$$\begin{aligned} d_X(\varphi_{t+T_1}(x), \varphi_t(x_2)) &\leq d_X(\varphi_{t+T_1}(x), \varphi_{t+T_1}(\hat{w})) + d_X(\varphi_{t+T_1}(\hat{w}), \varphi_t(x_2)) \\ &= d_X(\varphi_{t+T_1}(x), \varphi_{t+T_1}(\hat{w})) + d_X(\varphi_t(w), \varphi_t(x_2)) < 5\varepsilon. \end{aligned}$$

Furthermore,

$$\begin{aligned} \left| \frac{T - (T_1 + T_2)}{2} - \ln(1 + us) \right| &\leq \left| \frac{T - (T_1 + T_2)}{2} - \ln(1 + \tilde{u}\tilde{s}) \right| + |\ln(1 + \tilde{u}\tilde{s}) - \ln(1 + us)| \\ &\leq 5|\tilde{u}\tilde{s}|e^{-T_1-T_2} + 3|\tilde{u}\tilde{s} - us| = 3|us|(e^{-T_1} + e^{-T_2}) + 8|us|e^{-T_1-T_2}, \end{aligned}$$

due to (2.10). Finally, since $c_u a_{-T_1} = a_{-T_1} c_{ue^{-T_1}}$ and x_1 is T_1 -periodic, we obtain

$$x = \Gamma g_1 c_u a_{-T_1} c_\sigma b_\eta = \Gamma g_1 a_{-T_1} c_{ue^{-T_1}} c_\sigma b_\eta = \Gamma g_1 c_{ue^{-T_1+\sigma}} b_\eta,$$

where

$$|\sigma| < 2|\tilde{u}|e^{-T_1-T_2} < 2|u|e^{-T_1-T_2}$$

and

$$|\eta - s| < |\eta - \tilde{s}| + |\tilde{s} - s| < 2s^2|u| + 2|s|e^{-T_1-T_2}$$

which completes the proof. \square

2.3 Some auxiliary results

Lemma 2.4. *For every $\varepsilon > 0$ there is $\delta > 0$ with the following property. If $d_{\text{PSL}(2, \mathbb{R})}(g, h) < \delta$ then there are*

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

such that $g = \pi(G)$, $h = \pi(H)$ and $|g_{11} - h_{11}| + |g_{12} - h_{12}| + |g_{21} - h_{21}| + |g_{22} - h_{22}| < \varepsilon$.

See Lemma 2.17 (b) in [12] for a proof. Using the decomposition $g = c_u b_s a_\tau$, we have the following result.

Lemma 2.5. *If $s_1, s_2, s_3, u_1, u_2 \in]-\frac{1}{6}, \frac{1}{6}[$, then $b_{s_1} c_{u_1} b_{s_2} c_{u_2} b_{s_3} = c_u b_s a_\tau$ for*

$$\begin{aligned} u &= u_1 + u_2 + \frac{u_1 u_2 s_2 - (u_1 + u_2) \rho}{1 + \rho}, \\ s &= s_1 + s_2 + s_3 + \rho((2 + \rho)s_3 + s_1 + s_2) + u_1 s_1 s_2 (1 + \rho), \\ \tau &= 2 \ln(1 + \rho), \end{aligned}$$

where

$$\rho = u_2(s_1 + s_2) + u_1 s_1 (1 + u_2 s_2).$$

Lemma 2.6. *Let $\varepsilon \in]0, \min\{\frac{1}{2}, \frac{\sigma_0}{12}\}[$ and $x, x_1, x_2 \in X$ be given. If $x_j \in \mathcal{P}_\varepsilon(x)$ and $x_j = (u_j, s_j)_x, j = 1, 2$, then there is $\tau \in \mathbb{R}$ such that $\varphi_\tau(x_2) \in \mathcal{P}_{3\varepsilon}(x_1)$. Furthermore, $\varphi_\tau(x_2) = (u, s)_{x_1}$ satisfies*

$$|u - (u_2 - u_1)| < 8\varepsilon^3, \quad |s - (s_2 - s_1)| < 8\varepsilon^3, \quad |\tau| < 8\varepsilon^2,$$

and

$$|us - (u_2 - u_1)(s_2 - s_1)| = |s_1 s_2| (u_2 - u_1)^2.$$

Proof. Write $x = \Gamma g, x_1 = \Gamma g_1, x_2 = \Gamma g_2$ for $g, g_1, g_2 \in \mathbf{G}$. By assumption, $\Gamma g_j = \Gamma g c_{u_j} b_{s_j}$ for $j = 1, 2$. Then $\Gamma g_1 b_{-s_1} c_{u_2 - u_1} b_{s_2} = \Gamma g_2$, and u, s, τ are the numbers satisfying the decomposition $b_{-s_1} c_{u_2 - u_1} b_{s_2} = c_u b_s a_{-\tau}$. \square

Lemma 2.7. *Let $\varepsilon, T, T' > 0$. If*

$$d_X(\varphi_t(x_1), \varphi_t(x_2)) < \varepsilon \quad \text{for all } t \in [0, \min\{T, T'\}],$$

then

$$d_X(\varphi_t(x_1), \varphi_t(x_2)) < \varepsilon + \sqrt{2} |T - T'| \quad \text{for all } t \in [0, \max\{T, T'\}].$$

Proof. Write $x_1 = \Gamma g_1$ and $x_2 = \Gamma g_2$. Without loss of generality, we can assume that $T \leq T'$. By assumption, for $t \in [0, T]$,

$$d_X(\varphi_t(x_1), \varphi_t(x_2)) < \varepsilon \leq \varepsilon + \sqrt{2} |T - T'|.$$

For $t \in [T, \max\{T, T'\}] = [T, T']$,

$$\begin{aligned} d_X(\varphi_t(x_1), \varphi_t(x_2)) &\leq d_X(\varphi_t(x_1), \varphi_T(x_1)) + d_X(\varphi_T(x_1), \varphi_T(x_2)) + d_X(\varphi_T(x_2), \varphi_t(x_2)) \\ &= d_X(\Gamma g_1 a_t, \Gamma g_1 a_T) + d_X(\varphi_T(x_1), \varphi_T(x_2)) + d_X(\Gamma g_2 a_t, \Gamma g_2 a_T) \\ &\leq \varepsilon + 2d_{\mathbf{G}}(a_t, a_T) = \varepsilon + \sqrt{2} |t - T| \leq \varepsilon + \sqrt{2} |T' - T|. \end{aligned}$$

\square

Any periodic orbit of the flow $(\varphi_t)_{t \in \mathbb{R}}$ never comes back to another point on the stable manifold or the unstable manifold of a point on it. This follows from the next result.

Lemma 2.8. *Assume that $x, y \in X$ are periodic points of the flow $(\varphi_t)_{t \in \mathbb{R}}$ with the same period. Then $y \notin W_X^s(x)$ and $y \notin W_X^u(x)$.*

Proof. Let x, y be T -periodic points and suppose on the contrary that $y \in W_X^s(x), y \neq x$. But then

$$d_X(x, y) = d_X(\varphi_{mT}(x), \varphi_{mT}(y)) \rightarrow 0 \quad \text{as } m \rightarrow \infty$$

which is impossible since $d_X(x, y) > 0$. The case of unstable manifold can be treated analogously. \square

Owing to the hyperbolicity two periodic orbits with similar periods cannot stay too close together without being identical; see [12] for a proof.

Lemma 2.9. *Let $X = \Gamma \backslash \text{PSL}(2, \mathbb{R})$ be compact. Then there is $\varepsilon_* > 0$ with the following property. If $\varepsilon \in]0, \varepsilon_*[$ and if $x_1, x_2 \in X$ are periodic points of $(\varphi_t)_{t \in \mathbb{R}}$ having the periods $T_1, T_2 > 0$ such that $|T_1 - T_2| \leq \sqrt{2}\varepsilon$ and*

$$d_X(\varphi_t(x_1), \varphi_t(x_2)) < \varepsilon \quad \text{for all } t \in [0, \min\{T_1, T_2\}],$$

then the orbits of x_1 and x_2 under $(\varphi_t)_{t \in \mathbb{R}}$ are identical.

3 Higher-order encounters

In this section, we provide rigorous definitions of ‘ L -encounters’ and ‘partner orbits’, then we give an inductive argument to prove that a periodic orbit involving an L -parallel encounter satisfying a certain condition has $(L - 1)! - 1$ genuine partner orbits, and we derive an estimate for the action difference.

3.1 Encounters and partner orbits

We continue denoting $\mathcal{X} = T^1(\Gamma \backslash \mathbb{H}^2)$ and $X = \Gamma \backslash \text{PSL}(2, \mathbb{R})$.

Definition 3.1 (Time reversal). *The time reversal map $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ is defined by*

$$\mathcal{T}(p, \xi) = (-p, \xi) \quad \text{for } (p, \xi) \in \mathcal{X}.$$

The respective time reversal map on X is determined by

$$\mathcal{T}(x) = \Gamma g d_\pi \quad \text{for } x = \Gamma g \in X,$$

where $d_\pi \in \text{PSL}(2, \mathbb{R})$ is the equivalence class of the matrix $D_\pi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \text{SL}(2, \mathbb{R})$.

It is obvious that

$$\varphi_t(\mathcal{T}(x)) = \mathcal{T}(\varphi_{-t}(x)) \quad \text{for } x \in X \quad \text{and } t \in \mathbb{R}.$$

Recall the number ε_* from Lemma 2.9.

Definition 3.2 (Orbit pair/Partner orbit). *Let $\varepsilon \in]0, \varepsilon_*[$ be given. Two given T -periodic orbit c and T' -periodic orbit c' of the flow $(\varphi_t)_{t \in \mathbb{R}}$ are called an ε -orbit pair if there are $L \geq 2, L \in \mathbb{Z}$ and two decompositions of $[0, T]$ and $[0, T']$: $0 = t_0 < \dots < t_L = T$ and $0 = t'_0 < \dots < t'_L = T'$, and a permutation $\sigma : \{0, 1, \dots, L-1\} \rightarrow \{0, 1, \dots, L-1\}$ such that for each $j \in \{0, \dots, L-1\}$, either*

$$d_X(\varphi_{t+t_j}(x), \varphi_{t+t'_{\sigma(j)}}(x')) < \varepsilon \quad \text{for all } t \in [0, t_{j+1} - t_j]$$

or

$$d_X(\varphi_{t+t_j}(x), \varphi_{t-t'_{\sigma(j)+1}}(\mathcal{T}(x'))) < \varepsilon \quad \text{for all } t \in [0, t_{j+1} - t_j]$$

holds for some $x \in c$ and $x' \in c'$. Then c' is called an ε -partner orbit of c and vice versa.

Roughly speaking, two periodic orbits are an ε -orbit pair if they are ε -close to each other in configuration space, not for the whole time, since otherwise they would be identical due to Lemma 2.9, but they decompose to the same number of parts and any part of one orbit is ε -close to some part of the other. The above definition is modified from [4]. In this paper we will use the following equivalent definition.

Definition 3.3 (Orbit pair/Partner orbit). *Let $\varepsilon \in]0, \varepsilon_*[$ be given. Two given T -periodic orbit c and T' -periodic orbit c' of the flow $(\varphi_t)_{t \in \mathbb{R}}$ are called an ε -orbit pair if there are a number $L \geq 2, L \in \mathbb{Z}$ and a permutation $P : \{1, \dots, L\} \rightarrow \{1, \dots, L\}$ satisfying the following conditions.*

(a) *There are $x_1, \dots, x_n \in c, x'_1, \dots, x'_L \in c'$ such that*

$$\begin{aligned} \varphi_{T_j}(x_j) &= \begin{cases} x_{j+1} & \text{if } j \neq L \\ x_1 & \text{if } j = L \end{cases} \quad \text{for } T_j > 0 \quad \text{and} \quad T = T_1 + \dots + T_L, \\ \varphi_{T'_{P(j)}}(x'_j) &= \begin{cases} x'_{P(j)+1} & \text{if } P(j) \neq L \\ x'_1 & \text{if } P(j) = L \end{cases} \quad \text{for } T'_j > 0 \quad \text{and} \quad T' = T'_1 + \dots + T'_L; \end{aligned}$$

(b) *For $j = 1, \dots, L$,*

$$\text{either } d_X(\varphi_t(x'_j), \varphi_t(x_{P(j)})) < \varepsilon \quad \text{or} \quad d_X(\varphi_{t-T'_{P(j)}}(\mathcal{T}(x'_j)), \varphi_t(x_{P(j)})) < \varepsilon$$

holds for all $t \in [0, \max\{T_{P(j)}, T'_{P(j)}\}]$.

Then the orbit c' is called an ε -partner orbit of the orbit c and vice versa.

We shall often skip the parameter ε and call an ε -orbit pair an *orbit pair* and call an ε -partner orbit a *partner orbit* or a *partner*. It is clear that any periodic orbit always has 2 trivial partner orbits, namely itself and its time reversal orbit. We do not count them as partner orbits.

Definition 3.4 (Encounter). *Let $\varepsilon \in]0, \frac{\varepsilon_*}{4}[$ and $L \in \mathbb{Z}, L \geq 2$ be given. We say that a periodic orbit c has an (L, ε) -encounter if there are $x \in X, x_1, \dots, x_L \in c$ such that for each $j \in \{1, \dots, L\}$,*

$$\text{either } x_j \in \mathcal{P}_\varepsilon(x) \quad \text{or} \quad \mathcal{T}(x_j) \in \mathcal{P}_\varepsilon(x).$$

If either $x_j \in \mathcal{P}_\varepsilon(x)$ holds for all $i = 1, \dots, L$ or $\mathcal{T}(x_j) \in \mathcal{P}_\varepsilon(x)$ holds for all $j = 1, \dots, L$ then the encounter is called *parallel encounter*; otherwise it is called *antiparallel encounter*. We call the points x_1, \dots, x_L *piercing points* (see Figure 6 (a) & (b) below).

The parameter ε will be often dropped if the radius of the Poincaré section is not important.

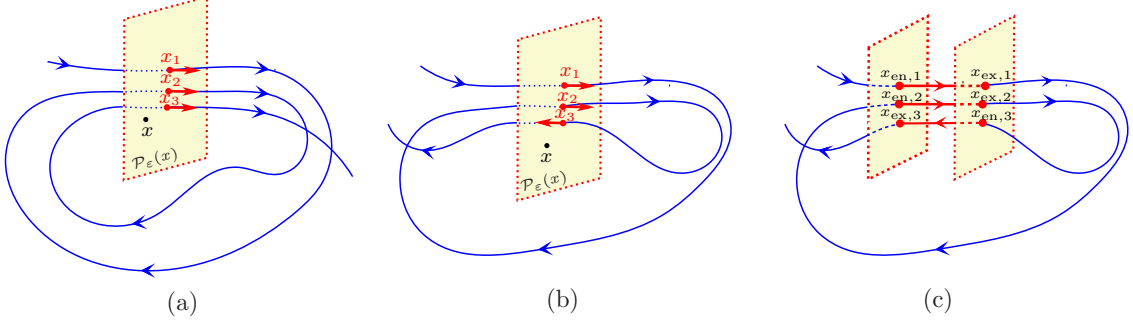


Figure 6: (a) & (b) Example of parallel and anti-parallel encounters; (c) Entrance and exit ports

Encounter duration

Given an (L, ε) -encounter with piercing points x_1, \dots, x_L . Owing to Lemma 2.6, we can assume that $x_i \in \mathcal{P}_\varepsilon(x_1)$ for $i = 2, \dots, L$. Note that either $d_X(x_j, x_i) < 4\varepsilon < \varepsilon_*$ or $d_X(\mathcal{T}(x_j), x_i) < 4\varepsilon < \varepsilon_*$ for any $i, j \in \{1, \dots, L\}$. By the continuity of the flow $(\varphi_t)_{t \in \mathbb{R}}$, there are L orbit stretches of length t_{enc} through x_1, \dots, x_L which remain 4ε -close to each other (up to time reversal), and we call these stretches the *encounter region*, and t_{enc} is called the *encounter duration*. We are going to evaluate t_{enc} . First, we consider a 2-parallel encounter. Let c be a T -periodic orbit with a 2-parallel encounter. Assume that $x, y \in c$ such that $y \in \mathcal{P}_\varepsilon(x)$ for $y = (u, s)_x$. Then

$$d_X(x, y) = d_X(\Gamma g, \Gamma g c_u b_s) \leq d_G(e, c_u b_s) \leq |u| + |s| < 2\varepsilon,$$

where $g \in G, \Gamma g = x$. Using $b_s a_t = a_t b_{s e^{-t}}$ and $c_u a_t = a_t c_{u e^t}$, we deduce that

$$\varphi_t(y) = \Gamma(g a_t)(c_{u e^t} b_{s e^{-t}}) \quad \text{for all } t \in \mathbb{R}.$$

Then $\varphi_t(y) \in \mathcal{P}_\varepsilon(\varphi_t(x))$ if and only if $|u|e^t < \varepsilon$ and $|s|e^{-t} < \varepsilon$. Note that $us \neq 0$ since otherwise $y \in W_X^s(x)$ or $y \in W_X^u(x)$ which contradicts Lemma 2.8. So, $\varphi_t(y) \in \mathcal{P}_\varepsilon(\varphi_t(x))$ for $-\ln\left(\frac{\varepsilon}{|s|}\right) < t < \ln\left(\frac{\varepsilon}{|u|}\right)$. Denote $t_s = \ln\left(\frac{\varepsilon}{|s|}\right)$ and $t_u = \ln\left(\frac{\varepsilon}{|u|}\right)$. Then for $t \in]-t_s, t_u[$, $\varphi_t(y) \in \mathcal{P}_\varepsilon(\varphi_t(x))$, $\varphi_t(y) = (u e^t, s e^{-t})_{\varphi_t(x)}$ and hence $d_X(\varphi_t(x), \varphi_t(y)) < 2\varepsilon$. The encounter duration is thus given by

$$t_{\text{enc}} = t_s + t_u = \ln\left(\frac{\varepsilon}{|s|}\right) + \ln\left(\frac{\varepsilon}{|u|}\right) = \ln\left(\frac{\varepsilon^2}{|us|}\right).$$

We see that t_s is the duration the flow can go backward and t_u is the duration the flow can go toward before leaving the encounter region. Next, for an (L, ε) -parallel encounter with piercing points $x_j \in \mathcal{P}_\varepsilon(x_1)$, $x_j = (u_j, s_j)_{x_1}$ for $j = 2, \dots, L$, we define

$$t_s = \min_{2 \leq j \leq L} \left\{ \ln \left(\frac{\varepsilon}{|u_j|} \right) \right\} \quad \text{and} \quad t_u = \min_{2 \leq j \leq L} \left\{ \ln \left(\frac{\varepsilon}{|s_j|} \right) \right\}. \quad (3.1)$$

The encounter duration is given by

$$t_{\text{enc}} = t_s + t_u = \ln \left(\frac{\varepsilon^2}{us} \right), \quad (3.2)$$

where $u = \max_{2 \leq j \leq L} \{|u_j|\}$, $s = \max_{2 \leq j \leq L} \{|s_j|\}$. For antiparallel encounters, the argument is similar and we also have (3.2) for the encounter duration; see [18] for similar results.

Definition 3.5 (Entrance port/Exit port). *Given an (L, ε) -encounter with piercing points x_1, \dots, x_L . For $j = 1, \dots, L$, we define the entrance port and the exit port of the j th orbit stretch by*

$$x_{\text{en},j} = \begin{cases} \varphi_{-t_s}(x_j) & \text{if } x_j \in \mathcal{P}_\varepsilon(x_1) \\ \varphi_{-t_u}(x_j) & \text{if } \mathcal{T}(x_j) \in \mathcal{P}_\varepsilon(x_1) \end{cases}$$

and

$$x_{\text{ex},j} = \begin{cases} \varphi_{t_u}(x_j) & \text{if } x_j \in \mathcal{P}_\varepsilon(x_1) \\ \varphi_{t_s}(x_j) & \text{if } \mathcal{T}(x_j) \in \mathcal{P}_\varepsilon(x_1), \end{cases}$$

respectively; recall t_s and t_u in (3.1).

We see that the j th stretch enters the encounter region through the entrance port $x_{\text{en},j}$ and leaves it through the exit port $x_{\text{ex},j}$ (see Figure 6 (c)).

Example 3.1. Assume that a periodic orbit of the geodesic flow on $T^1(\Gamma \backslash \mathbb{H}^2)$ crosses itself in configuration space at an angle θ such that $|\phi| < \min\{1/6, \varepsilon_*/9\}$ for $\phi = \pi - \theta$; see Figure 1. Then it has a unique partner orbit (see Theorem 3.15 in Part I [12]). Denote $\varepsilon = \frac{3}{2} \sin(\phi/2) < \frac{\varepsilon_*}{12} = \varrho$. Recall $x = \Gamma g, y = \Gamma h, y' = \mathcal{T}(y) = \Gamma h'$.

(i) The original orbit has a 2-antiparallel encounter. Indeed, by the proof of Theorem 3.5 in Part I, $x = \Gamma g = \Gamma h' d_\theta = \Gamma h' a_{-\tau} c_{-u} b_{-s}$, where $d_\theta \in \text{PSL}(2, \mathbb{R})$ is the equivalence class of $D_\theta = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \in \text{SL}(2, \mathbb{R})$, $\tau = -2 \ln(\cos(\phi/2))$, $s = \tan(\phi/2)$, $u = -\sin(\phi/2) \cos(\phi/2)$. This means that $x \in \mathcal{P}_\varrho(\tilde{y})$ for $x = (-u, -s)_{\tilde{y}}$ with $\tilde{y} = \varphi_{-\tau}(y')$. Consider the orbit c of $x' = \mathcal{T}(x)$. It follows from $x', \tilde{y} \in c$ and $\mathcal{T}(x') = x \in \mathcal{P}_\varrho(\tilde{y})$ that the orbit c has a 2-antiparallel encounter.

(ii) The partner orbit is a 6ε -partner. To see this, put $x_1 = x, x_2 = y, x'_1 = w, x'_2 = \varphi_{T_2}(w), T'_1 = T_2, T'_2 = T' - T'_1$. We have $\varphi_{T_1}(x_1) = x_2, \varphi_{T_2}(x_2) = x_1$ as well as $\varphi_{T'_1}(x'_1) = x'_2, \varphi_{T'_2}(x'_2) = x'_1$. Furthermore, it was shown in Part I that

$$d_X(\varphi_t(x_{P(1)}), \varphi_{t-T'_{P(1)}}(\mathcal{T}x'_1)) = d_X(\varphi_t(x_2), \varphi_t(\mathcal{T}x'_2)) < 6\varepsilon \quad \text{for } t \in [0, T_2]$$

and

$$d_X(\varphi_t(x_{P(2)}), \varphi_t(x'_2)) = d_X(\varphi_t(x_1), \varphi_t(x'_2)) < 6\varepsilon \quad \text{for } t \in [0, T_{P(2)}],$$

here $P = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Thus, the partner orbit is a 6ε -partner orbit.

(iii) The encounter duration

$$t_{\text{enc}} = \ln \left(\frac{\varrho^2}{\sin^2(\phi/2)} \right).$$

For more details, see Subsection 3.5 in Part I [12]. ◇

In the remaining part, encounters mean parallel encounters.

3.2 Number of partner orbits

Let c be a given periodic orbit with an L -encounter ($L \geq 2$). The orbit connects the j th entrance port and the j th exit port, $j = 1, \dots, L$. This can be described by the identical permutation $P_{\text{ori}} = \begin{pmatrix} 1 & 2 & \dots & L \\ 1 & 2 & \dots & L \end{pmatrix} = e$. We can connect the entrance ports and the exit ports by different ways to get different partner orbits. The way which entrance port is connected to which exit port can be expressed by a permutation $P \in S_L$. However, not all the permutations in S_L give connected partner orbits. A permutation P illustrated a partner orbit has to satisfy the condition that $P_{\text{loop}}P$ is a single cycle, where $P_{\text{loop}} = \begin{pmatrix} 1 & 2 & \dots & L-1 & L \\ 2 & 3 & \dots & L & 1 \end{pmatrix}$ is the *orbit loops permutation*, because it is a periodic orbit and hence returns to the first entrance port only after traversing all others. Recall that a permutation $P \in S_L$ is called a single cycle if P cannot be written as a product of shorter cycles; equivalently, $P^k(j) \neq j$ for all $j \in \{1, \dots, L\}$ and $k \in \{1, \dots, L-1\}$. For more details, see [18]. Note that we do not demand the permutation P to be a single cycle as in [18].

Lemma 3.1. *The number of permutations P in $S_L \setminus \{e\}$ such that $P_{\text{loop}}P$ are single cycles is $(L-1)! - 1$.*

Proof. It is well-known that the number of single cycles in S_L is $(L-1)!$. For every single cycle Q , we write $Q = P_{\text{loop}}(P_{\text{loop}}^{-1}Q)$. Then the permutation $P = P_{\text{loop}}^{-1}Q$ satisfies the condition that $P_{\text{loop}}P$ is a single cycle. Note that P_{loop} is a single cycle and the identity permutation e also satisfies the condition that $P_{\text{loop}}e$ is a single cycle. This completes the proof. □

Example 3.2. (i) For $L = 2$, there are $(2-1)! - 1 = 0$ partner orbits.

(ii) For $L = 3$, there is $(3-1)! - 1 = 1$ partner orbit which is illustrated by the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$; see Figure 7 (a).

(iii) For $L = 4$, there are $(4-1)! = 6$ single cycles in S_4 , namely:

$$(1 \ 2 \ 3 \ 4), (1 \ 2 \ 4 \ 3), (1 \ 3 \ 2 \ 4), (1 \ 3 \ 4 \ 2), (1 \ 4 \ 3 \ 2), (1 \ 4 \ 2 \ 3).$$

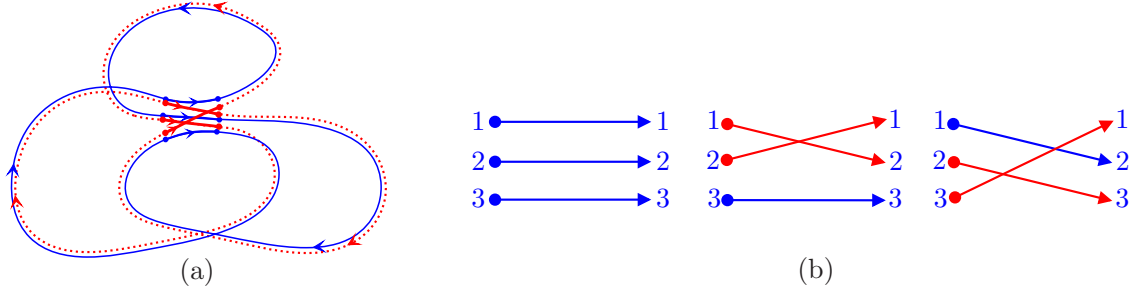


Figure 7: (a) Partner orbit of periodic orbit with 3-parallel encounter; (b) Interchange entrance and exit ports to create the partner orbit

The first one corresponds to the original orbit and therefore there are 5 partner orbits given by the following permutations:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}.$$

◇

3.3 3-encounters

In this subsection we will construct a partner orbit of a given orbit with a 3-encounter. Denote $P = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \in S_3$.

Theorem 3.1. *Let $\varepsilon \in]0, \frac{\varepsilon_*}{11}[$ and let c be a T -periodic orbit involving a $(3, \frac{\varepsilon}{3})$ -encounter with the following property:*

- (i) *there are $x_j \in c$, $x_j \in \mathcal{P}_{\frac{\varepsilon}{3}}(x)$ with $x_j = (u_j, s_j)_x$ for some $x \in X$, $j = 1, 2, 3$;*
- (ii) *$|u_j - u_i| > \frac{6}{5}(e^{-T_j} + e^{-T_i})$ and $|s_j - s_i| > \frac{24}{33}\varepsilon^3 + e^{-T_{j-1}} + e^{-T_{i-1}}$ for $j, i = 1, 2, 3$ and $j \neq i$, where $T_1, T_2, T_3 > 0$ are such that $T = T_1 + T_2 + T_3$, $\varphi_{T_1}(x_1) = x_2$, $\varphi_{T_2}(x_2) = x_3$, and $\varphi_{T_3}(x_3) = x_1$.*

Then the orbit c has a 11ε -partner c' whose period T' satisfies

$$\left| \frac{T' - T}{2} - \Delta S_3 \right| < 76\varepsilon^4 + 22\varepsilon^2 e^{-T_1} + 7\varepsilon^2 e^{-T_2} + 7\varepsilon^2 e^{-T_3}, \quad (3.3)$$

where

$$\Delta S_3 = \ln(1 + (u_3 - u_1)(s_3 - s_2)) + \ln(1 + (u_2 - u_1)(s_2 - s_1)).$$

Furthermore, the partner orbit also has a $(3, \varepsilon)$ -encounter. More precisely, there are $v_1, v_2, v_3 \in c'$ and $T'_1, T'_2, T'_3 > 0$ such that

(a) $\varphi_{T'_2}(v_1) = v_3$, $\varphi_{T'_3}(v_2) = v_1$, $\varphi_{T'_1}(v_3) = v_2$, and $T' = T'_1 + T'_2 + T'_3$;

(b) $v_j \in \mathcal{P}_\varepsilon(x)$ with $v_j = (u'_j, s'_j)_x$ satisfying

$$|u'_j - u_{P(j)}| < 23\varepsilon^3 + 6\varepsilon(e^{-T_1} + e^{-T_2} + e^{-T_3}), \quad (3.4a)$$

$$|s'_j - s_j| < 23\varepsilon^3 + 6\varepsilon(e^{-T_1} + e^{-T_2} + e^{-T_3}); \quad (3.4b)$$

(c) $|T'_j - T_j| < 22\varepsilon^2$;

(d) $d_X(\varphi_t(v_j), \varphi_t(x_{P(j)})) < 10\varepsilon$ for all $t \in [0, \max\{T_{P(j)}, T'_{P(j)}\}]$.

Proof. First we use Lemma 2.6 to write $\tilde{x}_j := \varphi_{\tilde{\tau}_j}(x_j) \in \mathcal{P}_\varepsilon(x_1)$ with $\tilde{x}_j = (\tilde{u}_j, \tilde{s}_j)_{x_1}$ and

$$|\tilde{u}_j - (u_j - u_1)| < \frac{1}{3}\varepsilon^3, \quad |\tilde{s}_j - (s_j - s_1)| < \frac{1}{3}\varepsilon^3, \quad |\tilde{u}_j \tilde{s}_j| < \frac{5}{9}\varepsilon^2, \quad |\tilde{\tau}_j| < \frac{8}{9}\varepsilon^2. \quad (3.5)$$

Denoting

$$\tilde{T}_1 = T_1 + \tilde{\tau}_2, \quad \tilde{T}_2 = T_2 - \tilde{\tau}_2 + \tilde{\tau}_3, \quad \text{and} \quad \tilde{T}_3 = T_3 - \tilde{\tau}_3, \quad (3.6)$$

we have

$$\varphi_{\tilde{T}_1}(x_1) = \tilde{x}_2, \quad \varphi_{\tilde{T}_2}(\tilde{x}_2) = \tilde{x}_3, \quad \varphi_{\tilde{T}_3}(\tilde{x}_3) = x_1, \quad (3.7)$$

and $T = \tilde{T}_1 + \tilde{T}_2 + \tilde{T}_3$ as well as

$$|T_1 - \tilde{T}_1| < \frac{8}{9}\varepsilon^2, \quad |T_2 - \tilde{T}_2| < 2\varepsilon^2, \quad \text{and} \quad |T_3 - \tilde{T}_3| < \frac{8}{9}\varepsilon^2. \quad (3.8)$$

Figure 7 illustrates steps of interchange ports in a 3-encounter to create a partner orbit.

Step 1: Write $x_j = \Gamma g_j$ and $\tilde{x}_j = \Gamma \tilde{g}_j$ for $g_j, \tilde{g}_j \in \mathbf{G}$, $j = 1, 2, 3$. By the above setting, we have $\tilde{x}_2 = \varphi_{\tilde{T}_1}(x_1) = (\tilde{u}_2, \tilde{s}_2)_{x_1} \in \mathcal{P}_\varepsilon(x_1)$. Apply the Anosov closing lemma I (Theorem 2.4) to obtain $y_2 = \Gamma h_2 = \Gamma g_1 c_{\sigma_2} b_{\eta_2} \in \mathcal{P}_{2\varepsilon}(x_1)$ and $\hat{T}_1 \in \mathbb{R}$ so that $\varphi_{\hat{T}_1}(y_2) = y_2$,

$$d_X(\varphi_t(x_1), \varphi_t(y_2)) < 4\varepsilon \quad \text{for all} \quad t \in [0, \tilde{T}_1], \quad (3.9)$$

$$\left| \frac{\hat{T}_1 - \tilde{T}_1}{2} - \ln(1 + \tilde{u}_2 \tilde{s}_2) \right| < 5|\tilde{u}_2 \tilde{s}_2|e^{-\tilde{T}_1}, \quad (3.10)$$

and

$$|\sigma_2| < 2\varepsilon e^{-\tilde{T}_1}, \quad |\eta_2 - \tilde{s}_2| < 2\varepsilon^3 + 2\varepsilon e^{-\tilde{T}_1}. \quad (3.11)$$

Then

$$|\hat{T}_1 - \tilde{T}_1| < 5|\tilde{u}_2 \tilde{s}_2| < 3\varepsilon^2 \quad (3.12)$$

yields

$$|\hat{T}_1 - T_1| \leq |\hat{T}_1 - \tilde{T}_1| + |\tilde{T}_1 - T_1| < 4\varepsilon^2 \quad (3.13)$$

due to (3.8). Furthermore,

$$|\eta_2 + s_1| \leq |\eta_2 - \tilde{s}_2| + |\tilde{s}_2 - (s_2 - s_1)| + |s_2| \leq 2\varepsilon^3 + 2\varepsilon e^{-\tilde{T}_1} + \frac{1}{3}\varepsilon^3 + \frac{1}{3}\varepsilon < \varepsilon$$

yields $|\eta_2| < \frac{4}{3}\varepsilon$. On the other hand, note that $\varphi_{\tilde{T}_2+\tilde{T}_3}(\tilde{x}_2) = x_1 = \Gamma\tilde{g}_2b_{-\tilde{s}_2}c_{-\tilde{u}_2} \in \mathcal{P}'_\varepsilon(\tilde{x}_2)$ and $\varphi_{\tilde{T}_2+\tilde{T}_3}(\tilde{x}_2) = (-\tilde{u}_2, -\tilde{s}_2)_{\tilde{x}_2}$. Then, by the Anosov closing lemma II (Theorem 2.5), there are $y_1 = \Gamma h_1 = \Gamma\tilde{g}_2b_{\eta_1}c_{\sigma_1} \in \mathcal{P}'_{2\varepsilon}(x_2)$ and $\hat{T}_{2,3} \in \mathbb{R}$ so that $\varphi_{\hat{T}_{2,3}}(y_1) = y_1$,

$$d_X(\varphi_t(y_1), \varphi_t(\tilde{x}_2)) < 4\varepsilon \quad \text{for all } t \in [0, \tilde{T}_2 + \tilde{T}_3], \quad (3.14)$$

$$\left| \frac{\hat{T}_{2,3} - (\tilde{T}_2 + \tilde{T}_3)}{2} \right| < 4|\tilde{u}_2\tilde{s}_2|e^{-\tilde{T}_2-\tilde{T}_3}, \quad (3.15)$$

and

$$|\sigma_1| < 2\varepsilon e^{-\tilde{T}_2-\tilde{T}_3}, \quad |\eta_1 + \tilde{s}_2| < 2\varepsilon e^{-\tilde{T}_2-\tilde{T}_3}. \quad (3.16)$$

Then

$$|\hat{T}_{2,3} - (\tilde{T}_2 + \tilde{T}_3)| < \frac{40}{9}\varepsilon^2 e^{-\tilde{T}_2-\tilde{T}_3} < \varepsilon^2 \quad (3.17)$$

and $|\eta_1| < \frac{4}{3}\varepsilon$.

Step 2: Construction of the partner orbit. We are going to ‘connect’ the orbits of y_1 and y_2 to get a new periodic orbit. We need to check the assumption of the connecting lemma (Lemma 2.3). Define $y_3 = \Gamma h_3 = \varphi_{\tilde{T}_2}(y_1)$ and recall $y_1 = \Gamma\tilde{g}_2b_{\eta_1}c_{\sigma_1}$, $\Gamma\tilde{g}_3 = \Gamma g_1c_{\tilde{u}_3}b_{\tilde{s}_3}$, and $\Gamma h_2 = \Gamma g_1c_{\tilde{u}_2}b_{\tilde{s}_2}$. This implies that

$$\begin{aligned} y_3 &= \Gamma\tilde{g}_2b_{\eta_1}c_{\sigma_1}a_{\tilde{T}_2} = \Gamma\tilde{g}_2a_{\tilde{T}_2}b_{\eta_1e^{-\tilde{T}_2}c_{\sigma_1e^{\tilde{T}_2}}} = \Gamma\tilde{g}_3b_{\eta_1e^{-\tilde{T}_2}c_{\sigma_1e^{\tilde{T}_2}}} \\ &= \Gamma g_1c_{\tilde{u}_3}b_{\tilde{s}_3+\eta_1e^{-\tilde{T}_2}c_{\sigma_1e^{\tilde{T}_2}}} = \Gamma h_2b_{-\eta_2}c_{\tilde{u}_3-\sigma_2}b_{\tilde{s}_3+\eta_1e^{-\tilde{T}_2}c_{\sigma_1e^{\tilde{T}_2}}}. \end{aligned}$$

Applying Lemma 2.5, we write $y_3 = \Gamma h_2c_{\tilde{u}_3}b_{\tilde{s}_3}a_{\tilde{\tau}_3}$ for

$$\begin{aligned} \tilde{u}_3 &= \tilde{u}_3 - \sigma_2 + \sigma_1e^{\tilde{T}_2} + \frac{1}{1+\tilde{\rho}_3}((\tilde{u}_3 - \sigma_2)\sigma_1e^{\tilde{T}_2}(\tilde{s}_3 + \eta_1e^{-\tilde{T}_2}) - (\tilde{u}_3 - \sigma_2 + \sigma_1e^{\tilde{T}_2})\tilde{\rho}_3), \\ \tilde{s}_3 &= -\eta_2 + \tilde{s}_3 + \eta_1e^{-\tilde{T}_2} + \tilde{\rho}_3(-\eta_2 + \tilde{s}_3 + \eta_1e^{-\tilde{T}_2}) - (\tilde{u}_3 - \sigma_2)\eta_2(\tilde{s}_3 + \eta_1e^{-\tilde{T}_2})(1+\tilde{\rho}_3), \\ \tilde{\tau}_3 &= 2\ln(1+\tilde{\rho}_3), \end{aligned}$$

where

$$\tilde{\rho}_3 = \sigma_1e^{\tilde{T}_2}(-\eta_2 + \tilde{s}_3 + \eta_1e^{-\tilde{T}_2}) - (\tilde{u}_3 - \sigma_2)\eta_2(1 + \sigma_1e^{\tilde{T}_2}(\tilde{s}_3 + \eta_1e^{-\tilde{T}_2})).$$

From $|\eta_1|, |\eta_2| < \frac{4\varepsilon}{3}$, (3.11), and (3.16), we thus have $|\tilde{\rho}_3| < 3\varepsilon^2$ and hence

$$|\tilde{\tau}_3| < 12\varepsilon^2, \quad |\tilde{u}_3 - \tilde{u}_3| < 7\varepsilon^3 + 2\varepsilon e^{-\tilde{T}_1} + 2\varepsilon e^{-\tilde{T}_3}, \quad |\tilde{s}_3 - (-\eta_2 + \tilde{s}_3)| < 16\varepsilon^3 + 2\varepsilon e^{-\tilde{T}_2}. \quad (3.18)$$

Therefore

$$|\tilde{u}_3 - (u_3 - u_1)| \leq |\tilde{u}_3 - (u_3 - u_1)| + |\tilde{u}_3 - \tilde{u}_3| \leq 8\varepsilon^3 + 2\varepsilon e^{-\tilde{T}_1} + 2\varepsilon e^{-\tilde{T}_3} \quad (3.19)$$

as well as

$$\begin{aligned} |\check{s}_3 - (s_3 - s_2)| &\leq |\check{s}_3 - (-\eta_2 + \tilde{s}_3)| + |\tilde{s}_2 - \eta_2| + |(s_2 - s_1) - \tilde{s}_2| + |\tilde{s}_3 - (s_3 - s_1)| \\ &< 19\varepsilon^3 + 2\varepsilon e^{-\tilde{T}_1} + 2\varepsilon e^{-\tilde{T}_2}, \end{aligned} \quad (3.20)$$

using (3.5) and (3.11), so that $|\check{u}_3| < \varepsilon$, $|\check{s}_3| < \varepsilon$, and hence

$$\check{y}_3 := \Gamma \check{h}_3 = \Gamma h_3 a_{-\tilde{\tau}_3} = \Gamma h_2 c_{\check{u}_3} b_{\check{s}_3} \in \mathcal{P}_\varepsilon(y_2). \quad (3.21)$$

Recall that $y_2 = \Gamma h_2$ is \hat{T}_1 -periodic and since $y_3 = \Gamma h_3$ is $\hat{T}_{2,3}$ -periodic, so is $\check{y}_3 = \varphi_{-\tilde{\tau}_3}(y_3)$. Apply Lemma 2.3 to obtain $z_3 = \Gamma h_2 c_{\check{u}_3 e^{-\hat{T}_1 + \sigma}} b_\eta \in \mathcal{P}_{2\varepsilon}(y_2)$ and $T' \in \mathbb{R}$ such that $\varphi_{T'}(z_3) = z_3$,

$$d_X(\varphi_t(z_3), \varphi_t(y_2)) < 5\varepsilon \quad \text{for all } t \in [0, \hat{T}_1], \quad (3.22)$$

$$d_X(\varphi_{t+\hat{T}_1}(z_3), \varphi_t(\check{y}_3)) < 5\varepsilon \quad \text{for all } t \in [0, \hat{T}_{2,3}], \quad (3.23)$$

$$\left| \frac{T' - (\hat{T}_1 + \hat{T}_{2,3})}{2} - \ln(1 + \check{u}_3 \check{s}_3) \right| < 5|\check{u}_3 \check{s}_3|(e^{-\hat{T}_1} + e^{-\hat{T}_{2,3}}), \quad (3.24)$$

and

$$|\sigma| < 2\varepsilon e^{-\hat{T}_1 - \hat{T}_{2,3}}, \quad |\eta - \check{s}_3| < 2\varepsilon^3 + 2\varepsilon e^{-\hat{T}_1 - \hat{T}_{2,3}} < 2\varepsilon^3 + \varepsilon e^{-T_2}. \quad (3.25)$$

Using (3.13) and (3.15), we rewrite (3.24) as

$$\left| \frac{T' - (\hat{T}_1 + \hat{T}_{2,3})}{2} - \ln(1 + \check{u}_3 \check{s}_3) \right| < 6|\check{u}_3 \check{s}_3|(e^{-\hat{T}_1} + e^{-\hat{T}_2 - \hat{T}_3}). \quad (3.26)$$

Step 3: Proof of (3.3). We are in a position to derive an estimate for the action difference. For, it follows from (3.26), (3.12), and (3.15) that

$$\begin{aligned} &\left| \frac{T' - T}{2} - \left(\ln(1 + \check{u}_3 \check{s}_3) + \ln(1 + \tilde{u}_2 \tilde{s}_2) \right) \right| \\ &\leq 6|\check{u}_3 \check{s}_3|(e^{-\hat{T}_1} + e^{-\hat{T}_2 - \hat{T}_3}) + 5|\tilde{u}_2 \tilde{s}_2|e^{-\hat{T}_1} + 4|\tilde{u}_2 \tilde{s}_2|e^{-\hat{T}_2 - \hat{T}_3} \\ &\leq 11\varepsilon^2 e^{-\hat{T}_1} + 10\varepsilon^2 e^{-\hat{T}_2 - \hat{T}_3}. \end{aligned} \quad (3.27)$$

Denote

$$\Delta S_3 = \ln(1 + (u_3 - u_1)(s_3 - s_2)) + \ln(1 + (u_2 - u_1)(s_2 - s_1))$$

and

$$\Delta S'_3 = \ln(1 + \check{u}_3 \check{s}_3) + \ln(1 + \tilde{s}_2 \tilde{s}_2).$$

Using the fact that $|\ln(1+x) - \ln(1+y)| < 3|x-y|$ for $|x|, |y| < \frac{1}{4}$, we have

$$\begin{aligned} |\Delta S'_3 - \Delta S_3| &\leq |\tilde{u}_2 \tilde{s}_2 - (u_2 - u_1)(s_2 - s_1)| + 3|\check{u}_3 \check{s}_3 - (u_3 - u_1)(s_3 - s_2)| \\ &< 76\varepsilon^2 + 10\varepsilon^2 e^{-\hat{T}_1} + 6\varepsilon^2 e^{-\hat{T}_2} + 6\varepsilon^2 e^{-\hat{T}_3} \end{aligned}$$

due to Lemma 2.6 and

$$\begin{aligned} |\check{u}_3 \check{s}_3 - (u_3 - u_1)(s_3 - s_2)| &\leq |\check{u}_3| |\check{s}_3 - (s_3 - s_2)| + |\check{u}_3 - (u_3 - u_1)| |s_3 - s_2| \\ &< 25\varepsilon^4 + \frac{10}{3} \varepsilon^2 e^{-\tilde{T}_1} + 2\varepsilon^2 e^{-\tilde{T}_2} + 2\varepsilon^2 e^{-\tilde{T}_3}, \end{aligned}$$

by (3.19) and (3.20). Therefore, it follows from (3.27) that

$$\begin{aligned} \left| \frac{T' - T}{2} - \Delta S_3 \right| &\leq 76\varepsilon^4 + 21\varepsilon^2 e^{-\tilde{T}_1} + \frac{13}{2} \varepsilon^2 e^{-\tilde{T}_2} + \frac{13}{2} \varepsilon^2 e^{-\tilde{T}_3} \\ &< 76\varepsilon^4 + 22\varepsilon^2 e^{-T_1} + 7\varepsilon^2 e^{-T_2} + 7\varepsilon^2 e^{-T_3}, \end{aligned}$$

which is (3.3).

Step 4: Definition of v_1, v_2, v_3 and proof of (3.4). In what follows we often use the fact that

$$|\eta_1| < \frac{4}{3} \varepsilon, |\eta_2| < \frac{4}{3} \varepsilon, |\sigma_1| < 2\varepsilon e^{-\tilde{T}_2 - \tilde{T}_3}, |\sigma_2| < 2\varepsilon e^{-\tilde{T}_1}, |\eta| < 2\varepsilon, |\sigma| < 2\varepsilon e^{-\tilde{T}_1 - \tilde{T}_2, 3}.$$

Step 4.1: Definition of v_3 . Recall that $\Gamma g_1 = \Gamma g_{c_{u_1} b_{s_1}}, \Gamma h_2 = \Gamma g_{c_{\sigma_2} b_{\eta_2}}, z_3 = \Gamma h_2 c_{\check{u}_3 e^{-\tilde{T}_1 + \sigma}} b_\eta$. Then using Lemma 2.5, we can write

$$z_3 = \Gamma h_2 c_{\check{u}_3 e^{-\tilde{T}_1 + \sigma}} b_\eta = \Gamma g_{c_{\sigma_2} b_{\eta_2} c_{\check{u}_3 e^{-\tilde{T}_1 + \sigma}}} b_\eta = \Gamma g_{c_{u_1} b_{s_1} c_{\sigma_2} b_{\eta_2} c_{\check{u}_3 e^{-\tilde{T}_1 + \sigma}}} b_\eta = \Gamma g_{c_{u'_3} b_{s'_3} a_{\tau'_3}}$$

and after a short estimate, we have

$$|\tau'_3| < \varepsilon^2, \quad |s'_3 - (s_1 + \eta_2 + \eta)| < \varepsilon e^{-T_1}, \quad |u'_3 - u_1| < 4\varepsilon e^{-T_1}.$$

Therefore, it follows from (3.25), (3.20), and (3.5) that

$$\begin{aligned} |s'_3 - s_3| &\leq |s'_3 - (s_1 + \eta_2 + \eta)| + |\eta - \check{s}_3| + |\check{s}_3 - (-\eta_2 + \tilde{s}_3)| + |\tilde{s}_3 - (s_3 - s_1)| \\ &< 23\varepsilon^3 + 5\varepsilon e^{-T_1} + 4\varepsilon e^{-T_2} \end{aligned}$$

and hence $|u'_3| < \varepsilon, |s'_3| < \varepsilon$. Defining $v_3 := \varphi_{-\tau'_3}(z_3)$, we have proven that $v_3 = \Gamma g_{c_{u'_3} b_{s'_3}} \in \mathcal{P}_\varepsilon(x)$ and $v_3 = (u'_3, s'_3)$ satisfies (3.68). Step 4.2: Definition of v_2 . Defining

$$z_2 := \varphi_{\hat{T}_1}(z_3) \tag{3.28}$$

and using Lemma 2.5, we write

$$\begin{aligned} z_2 &= \Gamma h_2 c_{\check{u}_3 + \sigma e^{\hat{T}_1}} b_{\eta e^{-\hat{T}_1}} = \Gamma g_{c_{\sigma_2} b_{\eta_2} c_{\check{u}_3 + \sigma e^{\hat{T}_1}}} b_{\eta e^{-\hat{T}_1}} \\ &= \Gamma g_{c_{u_1} b_{s_1} c_{\sigma_2} b_{\eta_2} c_{\check{u}_3 + \sigma e^{\hat{T}_1}}} b_{\eta e^{-\hat{T}_1}} = \Gamma g_{c_{u'_2} b_{s'_2} a_{\tau'_2}}, \end{aligned}$$

and after a short calculation, we obtain

$$|\tau'_2| < 5\varepsilon^2, \quad |s'_2 - (s_1 + \eta_2)| < 2\varepsilon^3 + 2\varepsilon e^{-\hat{T}_1}, \quad |u'_2 - (u_1 + \check{u}_3)| < \varepsilon^3 + 3\varepsilon e^{-T_1} + \varepsilon e^{-T_3}.$$

Together with (3.11), (3.5), and (3.19), this yields

$$|s'_2 - s_2| < 5\varepsilon^3 + 5\varepsilon e^{-T_1} \quad \text{as well as} \quad |u'_2 - u_3| < 13\varepsilon^3 + 6\varepsilon e^{-T_1} + 4\varepsilon e^{-T_3},$$

and therefore $|u'_2| < \varepsilon, |s'_2| < \varepsilon$. Defining $v_2 = \varphi_{-\tau'_2}(z_2)$, we have proven that $v_2 = \Gamma g c_{u'_2} b_{s'_2} \in \mathcal{P}_\varepsilon(x)$ and $v_2 = (u'_2, s'_2)_x$ satisfies (3.4). Step 4.3: Definition of v_1 . First, recall that $z_3 = \Gamma h_2 c_{\tilde{u}_3 e^{-\hat{T}_1 + \sigma}} b_\eta$. Then, since $y_2 = \Gamma h_2$ is a \hat{T}_1 -periodic point and $\Gamma h_3 a_{-\tilde{\tau}_3} b_{-\tilde{s}_3} = \Gamma h_2 c_{\tilde{u}_3}$ obtained from (3.21), we have

$$z_3 = \Gamma h_2 c_{\tilde{u}_3 e^{-\hat{T}_1 + \sigma}} b_\eta = \Gamma h_2 c_{\tilde{\tau}_3} a_{-\hat{T}_1} c_\sigma b_\eta = \Gamma h_3 a_{-\tilde{\tau}_3} b_{-\tilde{u}_3} a_{-\hat{T}_1} c_\sigma b_\eta. \quad (3.29)$$

Put $\hat{T}_3 = \hat{T}_{2,3} - \tilde{T}_2 + \tilde{\tau}_3$, $\hat{T}_{1,3} = \hat{T}_1 + \hat{T}_3$ and define

$$z_1 := \varphi_{\hat{T}_3}(z_2). \quad (3.30)$$

Due to $z_2 = \varphi_{\hat{T}_1}(z_3)$ and (3.29), it follows that

$$z_1 = \varphi_{\hat{T}_{1,3}}(z_3) = \Gamma h_3 a_{\hat{T}_{2,3} - \tilde{T}_2} b_{-\tilde{u}_3 e^{-\hat{T}_3} c_{\sigma e^{\hat{T}_{1,3}}} b_{\eta e^{-\hat{T}_{1,3}}} = \Gamma h_3 a_{-\tilde{T}_2} b_{-\tilde{u}_3 e^{-\hat{T}_3} c_{\sigma e^{\hat{T}_{1,3}}} b_{\eta e^{-\hat{T}_{1,3}}};$$

recall that $y_3 = \Gamma h_3$ is $\hat{T}_{2,3}$ -periodic. Using $y_3 = \Gamma h_3 = \varphi_{\tilde{T}_2}(y_1)$, $y_1 = \Gamma h_1 = \Gamma \tilde{g}_2 b_{\eta_1} c_{\sigma_1}$ and applying Lemma 2.5, we can write

$$\begin{aligned} z_1 &= \Gamma h_1 b_{-\tilde{u}_3 e^{-\hat{T}_3} c_{\sigma e^{\hat{T}_{1,3}}} b_{\eta e^{-\hat{T}_{1,3}}} = \Gamma g_2 a_{\tilde{\tau}_2} b_{\eta_1} c_{\sigma_1} b_{-\tilde{s}_3 e^{-\hat{T}_3} c_{\sigma e^{\hat{T}_{1,3}}} b_{\eta e^{-\hat{T}_{1,3}}} \\ &= \Gamma g c_{u_2} b_{s_2 + \eta_1 e^{\tilde{\tau}_2} c_{\sigma_1 e^{-\tilde{\tau}_2}} b_{-\tilde{s}_3 e^{-\hat{T}_3 + \tilde{\tau}_2} c_{\sigma e^{\hat{T}_{1,3} - \tilde{\tau}_2}} b_{\eta e^{-\hat{T}_{1,3} + \tilde{\tau}_2}} a_{\tilde{\tau}_2} \\ &= \Gamma g c_{u'_1} b_{s'_1} a_{\tau'_1}, \end{aligned}$$

with u'_1, s'_1, τ'_1 satisfy

$$|\tau'_1| < 2\varepsilon^2, \quad |u'_1 - u_2| \leq \varepsilon^3 + 3\varepsilon e^{-T_2}, \quad |s'_1 - (s_2 - \eta_1)| \leq 3\varepsilon^3 + 3\varepsilon e^{-T_3}$$

owing to

$$\eta_1 e^{\tilde{\tau}_2} = \eta_1 (1 + s_1(u_2 - u_1))^2 = \eta_1 + 2\eta_1 s_1(u_2 - u_1) + \eta_1 s_1^2(u_2 - u_1)^2.$$

Together with (3.5) and (3.16), this implies

$$|s'_1 - s_1| \leq |s'_1 - (s_2 - \eta_1)| + |\eta_1 + \tilde{s}_2| + |\tilde{s}_2 - (s_2 - s_1)| \leq 4\varepsilon^3 + 4\varepsilon e^{-T_3}. \quad (3.31)$$

Defining $v_1 := \varphi_{-\tau'_1}(z_1) = \Gamma g c_{u'_1} b_{s'_1}$ leads to $v_1 \in \mathcal{P}_\varepsilon(x)$ with $v_1 = (u'_1, s'_1)_x$ satisfying (3.4) for $j = 1$.

Step 5: Proof of (a)&(c). Recall from (3.28) and (3.30) that $z_2 = \varphi_{\hat{T}_1}(z_3)$ and $z_1 = \varphi_{\hat{T}_3}(z_2)$. Letting $\hat{T}_2 = T' - (\hat{T}_1 + \hat{T}_3)$, we have $\varphi_{\hat{T}_2}(z_2) = z_3$. From $v_j = \varphi_{-\tau'_j}(z_j)$, $j = 1, 2, 3$, we define

$$T'_1 = \hat{T}_1 + \tau'_3 - \tau'_2, \quad T'_2 = \hat{T}_2 - \tau'_3 + \tau'_1, \quad T'_3 = \hat{T}_3 + \tau'_2 - \tau'_1 \quad (3.32)$$

to obtain $T' = T'_1 + T'_2 + T'_3$ and

$$\varphi_{T'_2}(v'_1) = v'_3, \quad \varphi_{T'_3}(v'_2) = v'_1, \quad \varphi_{T'_1}(v'_3) = v'_2,$$

so (a) is shown. Now we show (c). It what follows we will use the following result several times:

$$|\tau'_1| < 2\varepsilon^2, \quad |\tau'_2| < 5\varepsilon^2, \quad |\tau'_3| < \varepsilon^2, \quad |\check{\tau}_3| < 12\varepsilon^2. \quad (3.33)$$

First, it follows from (3.32) that

$$|T'_1 - T_1| \leq |T'_1 - \hat{T}_1| + |\hat{T}_1 - T_1| \leq |\tau'_3| + |\tau'_2| + |\hat{T}_1 - T_1| \leq 10\varepsilon^2$$

due to (3.13). A short calculation shows that

$$T'_2 - T_2 = T' - (\hat{T}_1 + \hat{T}_{2,3}) + \tilde{T}_2 - T_2 - \check{\tau}_3 - \tau'_3 + \tau'_1.$$

Hence, by (3.8) and (3.33),

$$|T'_2 - T_2| \leq |T' - (\hat{T}_1 + \hat{T}_{2,3})| + |\tilde{T}_2 - T_2| + |\check{\tau}_3| + |\tau'_3| + |\tau'_1| < 22\varepsilon^2;$$

here we have used $|T' - (\hat{T}_1 + \hat{T}_{2,3})| < 5\varepsilon^2$ obtained from (3.26). Finally, recall that $\hat{T}_3 = \hat{T}_{2,3} - \tilde{T}_2 + \check{\tau}_3$. Then it follows from (3.32) that

$$|T'_3 - T_3| \leq |T'_3 - \hat{T}_3| + |\hat{T}_3 - T_3| \leq |\hat{T}_{2,3} - \tilde{T}_2 - \tilde{T}_3| + |T_3 - \tilde{T}_3| + |\check{\tau}_3| + |\tau'_2| + |\tau'_1| < 21\varepsilon^2,$$

using (3.17), (3.8), and (3.18). In summary, $|T'_j - T_j| < 22\varepsilon^2$ for $j = 1, 2, 3$ as was to be shown in (c).

Step 6: Proof of (d). It what follows we often use the fact that if $z, v \in X$ and $z = \varphi_\tau(v)$ for some $\tau \in \mathbb{R}$, then

$$d_X(\varphi_t(v), \varphi_t(z)) \leq d_G(a_\tau, e) < |\tau| \quad \text{for all } t \in \mathbb{R}.$$

This applies to $v_j = \varphi_{-\tau'_j}(z_j)$, $j = 1, 2, 3$. Step 6.1: For $j = 1$. For $t \in [0, \min\{\tilde{T}_1, \hat{T}_1\}]$,

$$\begin{aligned} d_X(\varphi_t(v_3), \varphi_t(x_1)) &\leq d_X(\varphi_t(v_3), \varphi_t(z_3)) + d_X(\varphi_t(z_3), \varphi_t(y_2)) + d_X(\varphi_t(y_2), \varphi_t(x_1)) \\ &\leq |\tau'_3| + 5\varepsilon + 4\varepsilon < 9\varepsilon + \varepsilon^2, \end{aligned}$$

due to (3.9) and (3.22). Therefore, since $|T_1 - \tilde{T}_1| < \varepsilon^2$, $|T'_1 - \hat{T}_1| < 6\varepsilon^2$, we have for all $t \in [0, \max\{T_1, T'_1\}]$:

$$d_X(\varphi_t(v_3), \varphi_t(x_1)) < 9\varepsilon + \varepsilon^2 + \sqrt{2} \max\{|T - \tilde{T}_1|, |T'_1 - \hat{T}_1|\} < 10\varepsilon,$$

using Lemma 2.7, and thus (d) is obtained for $j = 1$. Step 6.2: For $j = 2$. Recall that $z_2 = \varphi_{\hat{T}_1}(z_3)$, $y_3 = \varphi_{\tilde{T}_2}(y_2)$, $\tilde{x}_3 = \varphi_{\tilde{T}_2}(\tilde{x}_2)$, $\check{y}_3 = \varphi_{-\check{\tau}_3}(y_3)$, and $\tilde{x}_3 = \varphi_{\check{\tau}_3}(x_3)$. It follows from (3.14), (3.23), (3.5), and (3.33) that for $t \in [0, \tilde{T}_3]$,

$$\begin{aligned} d_X(\varphi_t(v_2), \varphi_t(x_3)) &< d_X(\varphi_t(v_2), \varphi_t(z_2)) + d_X(\varphi_t(z_2), \varphi_t(\check{y}_3)) + d_X(\varphi_t(\check{y}_3), \varphi_t(y_3)) \\ &\quad + d_X(\varphi_t(y_3), \varphi_t(\tilde{x}_3)) + d_X(\varphi_t(\tilde{x}_3), \varphi_t(x_3)) \\ &< |\tau'_2| + d_X(\varphi_{t+\hat{T}_1}(z_3), \varphi_t(\check{y}_3)) + |\check{\tau}_3| + d_X(\varphi_{t+\tilde{T}_2}(y_1), \varphi_{t+\tilde{T}_2}(\tilde{x}_2)) + |\check{\tau}_3| \end{aligned}$$

$$< 9\varepsilon + 18\varepsilon^2.$$

Owing to $|T_3 - \tilde{T}_3| < \varepsilon^2$ and $|T'_3 - T_3| < 22\varepsilon^2$, we apply Lemma 2.7 to obtain (d) for $j = 2$.

Step 6.3: For $j = 3$. The argument is analogous.

Step 7: *The distinction between the partner orbit and the original orbit.* We skip it and will prove it in the next theorem. \square

Remark 3.1. (a) According to Step 3 of the proof, the action difference between the orbit pair satisfies

$$\left| \frac{T' - T}{2} - \left(\ln(1 + \tilde{u}_3 \tilde{s}_3) + \ln(1 + \tilde{u}_2 \tilde{s}_2) \right) \right| \leq 7(|\tilde{u}_3 \tilde{s}_3| + |\tilde{u}_2 \tilde{s}_2|)(e^{-T_1} + e^{-T_2 - T_3}).$$

(b) The term $76\varepsilon^4$ in (3.3) arises from the coordinate changes $(\tilde{u}_3, \tilde{s}_3) \rightarrow (u_3 - u_1, s_3 - s_2)$ and $(\tilde{u}_2, \tilde{s}_2) \rightarrow (u_2 - u_1, s_2 - s_1)$; and it cannot be avoided. \diamond

3.4 L -encounters

Let c be a T -periodic orbit involving an L -encounter ($L \geq 3$). Without loss of generality, we assume that the encounter corresponds to the trivial permutation $e = \begin{pmatrix} 1 & 2 & \cdots & L \\ 1 & 2 & \cdots & L \end{pmatrix}$ and its orbit loops correspond to the permutation $P_{\text{loop}} = \begin{pmatrix} 1 & 2 & \cdots & L \\ 2 & 3 & \cdots & 1 \end{pmatrix}$; recall section 3.2. Let P be a permutation in S_L such that $P_{\text{loop}}P$ is a single cycle. In this subsection, we will construct the partner orbit given by P . We define the *sequence* $\{P_k\}_{k=0, \dots, L-3}$ generated by P as follows. Put $P_0 := P$ and for $k \in \{1, \dots, L-3\}$, define $P_k : \{1, 2, \dots, L\} \setminus \{2, \dots, k+1\} \rightarrow \{1, 2, \dots, L\} \setminus \{1, \dots, k\}$ recursively by

$$P_k(j) = \begin{cases} P_{k-1}(j) & \text{if } j \neq P_{k-1}^{-1}(k) \\ P_{k-1}(k+1) & \text{if } j = P_{k-1}^{-1}(k); \end{cases} \quad (3.34)$$

the respective orbit loops $P_{k, \text{loop}} : \{1, 2, \dots, L\} \setminus \{1, \dots, k\} \rightarrow \{1, 2, \dots, L\} \setminus \{2, \dots, k+1\}$ is defined by

$$P_{k, \text{loop}}(j) = \begin{cases} 1 & \text{if } j = L, \\ j+1 & \text{otherwise.} \end{cases}$$

Then $P_{k, \text{loop}}P_k$ is a permutation of the set $\{1, 2, \dots, L\} \setminus \{2, \dots, k+1\}$ and is a single cycle. Denote

$$\Delta d_{(L)} = 41\left(\frac{1}{3} + \cdots + \frac{1}{L}\right), \quad \Delta T_{(L)} = 14 + 78\left(\frac{1}{3^2} + \cdots + \frac{1}{L^2}\right), \quad (3.35)$$

$$\alpha_L = 6 + 468\left(\frac{1}{3^3} + \cdots + \frac{1}{L^3}\right), \quad \beta_L = 1 + 17\left(\frac{1}{3} + \cdots + \frac{1}{L}\right), \quad (3.36)$$

and recall ε_* from Lemma 2.9. The main result of this paper is the following.

Theorem 3.2. *For $\varepsilon \in]0, \frac{\varepsilon_*}{\Delta d_{(L)}}]$, assume that the T -periodic orbit c has an $(L, \frac{\varepsilon}{L})$ -encounter with the following property:*

(i) *there are $x_j \in c$, $x_j \in \mathcal{P}_{\frac{\varepsilon}{L}}(x)$, $x_j = (u_j, s_j)_x$ for $j = 1, \dots, L$ and some $x \in X$;*

(ii) $|u_j - u_i| > \frac{6}{5} (e^{-T_j} + e^{-T_i})$ and $|s_j - s_i| > \frac{24}{L^3} \varepsilon^3 + e^{-T_{j-1}} + e^{-T_{i-1}}$ for $j \neq i$; here $T_1, \dots, T_L > 0$, $T = T_1 + \dots + T_L$, $\varphi_{T_j}(x_j) = x_{j+1}$ for $j = 1, \dots, L-1$ and $\varphi_{T_L}(x_L) = x_1$.

Then for every $P \in S_L$ such that $P_{\text{loop}}P$ is a single cycle, the orbit c has a $\Delta d_{(L)}\varepsilon$ -partner c' given by P with period T' satisfying

$$\left| \frac{T' - T}{2} - \Delta S_L \right| \leq \omega_L \varepsilon^4 + \kappa_L \varepsilon^2 (e^{-T_1} + \dots + e^{-T_L}), \quad (3.37)$$

where

$$\Delta S_L = \sum_{j=1}^{L-2} \ln(1 + (u_{j+1} - u_j)(s_{j+1} - s_1)) + \sum_{j=1}^{L-2} \ln(1 + (u_{P_{j-1}(j+1)} - u_j)(s_{P_{j-1}^{-1}(j)} - s_{j+1})),$$

$$\omega_L = 12 \left(\frac{1}{3^4} + \dots + \frac{1}{L^4} \right) + 720 \left(\frac{1}{3^3} + \dots + \frac{1}{L^3} \right) + \frac{21\alpha_L}{L} - 114, \quad (3.38)$$

$$\kappa_L = 118 \left(\frac{1}{3^2} + \dots + \frac{1}{L^2} \right) + 312 \left(\frac{1}{3} + \dots + \frac{1}{L} \right) + \frac{21\beta_L}{L} - 156. \quad (3.39)$$

Furthermore, the partner orbit c' has an $(L, \frac{3\varepsilon}{L})$ -encounter. More precisely, there are $v_1, \dots, v_L \in c'$, $T'_1, \dots, T'_L > 0$ such that for $j = 1, \dots, L$:

$$(a) \quad \varphi_{T'_{P(j)}}(v_j) = \begin{cases} v_{P(j)+1} & \text{if } P(j) \neq L \\ v_1 & \text{if } P(j) = L \end{cases} \quad \text{and} \quad T' = T'_1 + \dots + T'_L;$$

$$(b) \quad v_j \in \mathcal{P}_{\frac{3\varepsilon}{L}}(x) \text{ for } v_j = (u'_j, s'_j)_x \text{ satisfying}$$

$$|u'_j - u_{P(j)}| < \alpha_L \varepsilon^3 + \beta_L \varepsilon (e^{-T_1} + \dots + e^{-T_L}), \quad (3.40a)$$

$$|s'_j - s_j| < \alpha_L \varepsilon^3 + \beta_L \varepsilon (e^{-T_1} + \dots + e^{-T_L}); \quad (3.40b)$$

$$(c) \quad |T'_j - T_j| < \Delta T_{(L)} \varepsilon^2;$$

$$(d) \quad d_X(\varphi_t(x_{P(j)}), \varphi_t(v_j)) < \Delta d_{(L)} \varepsilon \quad \text{for all } t \in [0, \max\{T_{P(j)}, T'_{P(j)}\}].$$

Proof. Let $L \geq 3$ be an integer number. We see that $\Delta T_{(L)} = \Delta T_{(L-1)} + \frac{78}{L^2}$ and $\Delta d_{(L)} = \Delta d_{(L-1)} + \frac{41}{L}$. We will prove this theorem by induction. For $L = 3$, only one permutation $P = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ satisfies the assumption, and Theorem 3.1 proves this case; note that $\omega_3 > 76$, $\kappa_3 > 7$, $\alpha_3 > 23$, and $\beta_3 > 6$. Assume that the theorem is correct for $L = n - 1$ for $n \geq 5$, we prove that it is correct for $L = n$.

Let c be a periodic orbit with n -encounter illustrated by the trivial permutation and let $P \in S_n$ be a permutation such that $P_{\text{loop}}P$ is a single cycle. Suppose that there are $x \in X$, $x_1, \dots, x_n \in c$, $x_j = (u_j, s_j)_x$ satisfying the assumption. Using Lemma 2.6, write $\tilde{x}_j := \varphi_{\tilde{\tau}_j}(x_j) \in \mathcal{P}_{\frac{3\varepsilon}{n}}(x_1)$ with $\tilde{x}_j = (\tilde{u}_j, \tilde{s}_j)_{x_1}$, and

$$|\tilde{u}_j - (u_j - u_1)| < \frac{8}{n^3} \varepsilon^3, \quad |\tilde{s}_j - (s_j - s_1)| < \frac{8}{n^3} \varepsilon^3, \quad |\tilde{\tau}_j| < \frac{8}{n^2} \varepsilon^2, \quad (3.41)$$

$$|\tilde{u}_j \tilde{s}_j - (u_j - u_1)(s_j - s_1)| < \frac{4}{n^4} \varepsilon^4, \quad |\tilde{u}_j \tilde{s}_j| < \frac{5}{n^2} \varepsilon^2. \quad (3.42)$$

Denoting $\tilde{T}_1 = T_1 + \tilde{\tau}_2$, $\tilde{T}_j = T_j - \tilde{\tau}_j + \tilde{\tau}_{j+1}$, for $j = 2, \dots, n-1$, and $\tilde{T}_n = T_n - \tilde{\tau}_n$, we obtain $T = \tilde{T}_1 + \dots + \tilde{T}_n$,

$$\varphi_{\tilde{T}_j}(\tilde{x}_j) = \tilde{x}_{j+1} \quad \text{for } j = 1, \dots, n-1 \quad \text{and} \quad \varphi_{\tilde{T}_n}(\tilde{x}_n) = \tilde{x}_1 = x_1. \quad (3.43)$$

Furthermore,

$$|\tilde{T}_1 - T_1| = |\tilde{\tau}_2| < \frac{8}{n^2} \varepsilon^2, \quad |\tilde{T}_j - T_j| \leq \frac{12}{n^2} \varepsilon^2 \text{ for } j = 2, \dots, n-1, \quad |\tilde{T}_n - T_n| = |\tilde{\tau}_n| < \frac{8}{n^2} \varepsilon^2. \quad (3.44)$$

Figure 8 below illustrates the idea of the proof. The encounter of the original orbit corresponds the trivial permutation depicted in (a). In (b), we exchange the ports of the first two stretches to have two shorter periodic orbits c_1 and c_2 (Step 1 below). The longer one c_1 has $(n-1)$ -encounter. We construct the partner orbit c'_1 corresponding the permutation P_1 for the c_1 , expressed in (c) (Step 2). Finally, we connect two orbits to have a new partner orbit described in (d) (Step 3).

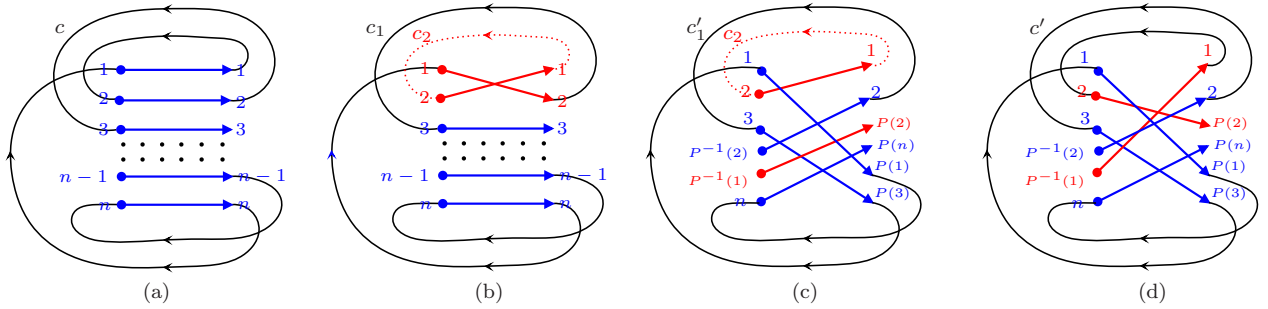


Figure 8: Inductive argument to construct partner orbits

Step 1: Reducing the encounter order. Write $x_j = \Gamma g_j$ and $\tilde{x}_j = \Gamma \tilde{g}_j$ for $g_j, \tilde{g}_j \in \mathbb{G}, j = 1, \dots, n$. By the above setting, $\tilde{x}_2 = \varphi_{\tilde{T}_1}(x_1) \in \mathcal{P}_{\frac{3\varepsilon}{n}}(x_1)$ with $\varphi_{\tilde{T}_1}(x_1) = (\tilde{u}_2, \tilde{s}_2)_{x_1}$. Note that assumption (ii) and (3.44) guarantee that $\tilde{T}_1 \geq 1$. Then by the Anosov closing lemma I, there are $\hat{T}_1 \in \mathbb{R}$ and $y_2 = \Gamma g_1 c_{\sigma_2} b_{\eta_2}$ so that $\varphi_{\hat{T}_1}(y_2) = y_2$,

$$|\eta_2 - \tilde{s}_2| < \frac{30}{n^3} \varepsilon^3 + \frac{6}{n} \varepsilon e^{-\tilde{T}_1}, \quad |\sigma_2| < 2|\tilde{u}_2| e^{-\tilde{T}_1} < \frac{6}{n} \varepsilon e^{-\tilde{T}_1}, \quad (3.45)$$

$$d_X(\varphi_t(x_1), \varphi_t(y_2)) < 2|\tilde{u}_2| + |\eta_2| \quad \text{for all } t \in [0, \hat{T}_1],$$

and

$$\left| \frac{\hat{T}_1 - \tilde{T}_1}{2} - \ln(1 + \tilde{u}_2 \tilde{s}_2) \right| \leq 5|\tilde{u}_2 \tilde{s}_2| e^{-\tilde{T}_1}.$$

Then $|\hat{T}_1 - \tilde{T}_1| < \frac{21}{n^2} \varepsilon^2$ implies that

$$|\hat{T}_1 - T_1| < |\hat{T}_1 - \tilde{T}_1| + |\tilde{T}_1 - T_1| < \frac{29}{n^2} \varepsilon^2 \quad (3.46)$$

due to (3.44). Furthermore, using (3.42), we have

$$|\ln(1 + \tilde{u}_2 \tilde{s}_2) - \ln(1 + (u_2 - u_1)(s_2 - s_1))| < 3|\tilde{u}_2 \tilde{s}_2 - (u_2 - u_1)(s_2 - s_1)| < \frac{12}{n^4} \varepsilon^4,$$

owing to $|\ln(1 + x) - \ln(1 + y)| < 3|x - y|$ for $x, y \in [0, \frac{1}{2}]$; and so,

$$\left| \frac{\hat{T}_1 - \tilde{T}_1}{2} - \ln(1 + (u_2 - u_1)(s_2 - s_1)) \right| < \frac{12}{n^4} \varepsilon^4 + \frac{26}{n^2} \varepsilon^2 e^{-T_1}. \quad (3.47)$$

It follows from (3.45) and (3.41) that

$$|\eta_2 + s_1| \leq |\eta_2 - \tilde{s}_2| + |\tilde{s}_2 - (s_2 - s_1)| + |s_2| < \frac{2}{n} \varepsilon, \quad (3.48)$$

whence $|\eta_2| < \frac{3}{n} \varepsilon$, and consequently

$$d_X(\varphi_t(x_1), \varphi_t(y_2)) < \frac{9}{n} \varepsilon \quad \text{for all } t \in [0, \tilde{T}_1]. \quad (3.49)$$

The orbit of y_2 is depicted by the dotted line c_2 in Figure 8 (b).

Similarly, $\varphi_{T-\tilde{T}_1}(\tilde{x}_2) = x_1 = \Gamma \tilde{g}_2 b_{-\tilde{s}_2} c_{-\tilde{u}_2} \in \mathcal{P}'_{\frac{3\varepsilon}{n}}(\tilde{x}_2)$ with $\varphi_{T-\tilde{T}_1}(\tilde{x}_2) = (-\tilde{s}_2, -\tilde{u}_2)'_{\tilde{x}_2}$. Applying the Anosov closing lemma II, there are $y_1 = \Gamma \tilde{g}_2 b_{\eta_1} c_{\sigma_1}$ and $\tilde{T} \in \mathbb{R}$ so that $\varphi_{\tilde{T}}(y_1) = y_1$,

$$\left| \frac{\tilde{T} - (T - \tilde{T}_1)}{2} \right| < 4|\tilde{u}_2 \tilde{s}_2| e^{-T+\tilde{T}_1} < \frac{1}{n^2} \varepsilon^2 e^{-T_2}, \quad (3.50)$$

$$d_X(\varphi_t(y_1), \varphi_t(\tilde{x}_2)) < 2|\tilde{u}_2| + |\eta_1| \quad \text{for all } t \in [0, T - \tilde{T}_1],$$

and

$$|\sigma_1| < \frac{6}{n} \varepsilon e^{-(\tilde{T}_2 + \dots + \tilde{T}_n)}, \quad |\tilde{s}_2 + \eta_1| < \frac{6}{n} \varepsilon e^{-(\tilde{T}_2 + \dots + \tilde{T}_n)}.$$

Then

$$|\sigma_1| < \frac{7}{n} \varepsilon e^{-(T_2 + \dots + T_n)}, \quad |\tilde{s}_2 + \eta_1| < \frac{7}{n} \varepsilon e^{-(T_2 + \dots + T_n)} \quad (3.51)$$

and (3.41) yield

$$|\eta_1| \leq |\tilde{s}_2 + \eta_1| + |\tilde{s}_2 - (s_2 - s_1)| + |s_2 - s_1| < \frac{3}{n} \varepsilon;$$

so that

$$d_X(\varphi_t(y_1), \varphi_t(\tilde{x}_2)) < \frac{9}{n} \varepsilon \quad \text{for all } t \in [0, T - \tilde{T}_1]. \quad (3.52)$$

The orbit of y_1 is expressed by the solid line c_1 in Figure 8 (b).

In order to apply the inductive assumption, it is necessary to verify that the orbit c_1 through y_1 has an $(n-1)$ -encounter. Indeed, recall that $y_1 = \Gamma \tilde{g}_2 b_{\eta_1} c_{\sigma_1} = \Gamma g_1 c_{\tilde{u}_2} b_{\tilde{s}_2} b_{\eta_1} c_{\sigma_1} = \Gamma g c_{u_1} b_{s_1} c_{\tilde{u}_2} b_{\tilde{s}_2 + \eta_1} c_{\sigma_1}$. Apply Lemma 2.5 to write $y_1 = \Gamma g c_{\tilde{u}_1} b_{\tilde{s}_1} a_{\tilde{\tau}_1}$ for

$$\begin{aligned}\tilde{u}_1 &= u_1 + \tilde{u}_2 + \sigma_1 + \frac{1}{1 + \check{\rho}_1} (\tilde{u}_2 \sigma_1 (\tilde{s}_2 + \eta_1) - (\tilde{u}_2 + \sigma_1) \check{\rho}_1), \\ \check{s}_1 &= s_1 + \tilde{s}_2 + \eta_1 + \check{\rho}_1 (s_1 + \tilde{s}_2 + \eta_1) + \tilde{u}_2 s_1 (\tilde{s}_2 + \eta_1) (1 + \check{\rho}_1), \\ \check{\tau}_1 &= 2 \ln(1 + \check{\rho}_1),\end{aligned}$$

where

$$\check{\rho}_1 = \sigma_1 (s_1 + \tilde{s}_2 + \eta_1) + \tilde{u}_2 s_1 (1 + \sigma_1 (\tilde{s}_2 + \eta_1)).$$

Using (3.51), we have $|\check{\rho}_1| < \frac{4}{n^2} \varepsilon^2$. This implies that

$$\begin{aligned}|\check{u}_1 - u_2| &\leq |\check{u}_1 - (u_1 + \tilde{u}_2)| + |\tilde{u}_2 - (u_2 - u_1)| \\ &< \frac{28}{n^3} \varepsilon^3 + \frac{8}{n} \varepsilon e^{-(T_2 + \dots + T_n)},\end{aligned}\tag{3.53}$$

$$|\check{s}_1 - s_1| < \frac{2}{n^3} \varepsilon^3,\tag{3.54}$$

$$|\check{\tau}_1| < \frac{16}{n^2} \varepsilon^2.\tag{3.55}$$

For $j = 3, \dots, n$, define $y_j := \varphi_{-\tilde{\tau}_2 + T_2 + \dots + T_{j-1}}(y_1)$. Then, using $\varphi_{T_2 + \dots + T_{j-1}}(x_2) = x_j = (u_j, s_j)_x$ and Lemma 2.5, we write

$$\begin{aligned}y_j &= \Gamma \tilde{g}_2 a_{-\tilde{\tau}_2 + T_2 + \dots + T_{j-1}} b_{\eta_1 e^{-(\tilde{\tau}_2 + T_2 + \dots + T_{j-1})} C_{\sigma_1} e^{-\tilde{\tau}_2 + T_2 + \dots + T_{j-1}}} \\ &= \Gamma g_j b_{\eta_1 e^{-(\tilde{\tau}_2 + T_2 + \dots + T_{j-1})} C_{\sigma_1} e^{-\tilde{\tau}_2 + T_2 + \dots + T_{j-1}}} \\ &= \Gamma g c_{u_j} b_{s_j + \eta_1 e^{-(\tilde{\tau}_2 + T_2 + \dots + T_{j-1})} C_{\sigma_1} e^{-\tilde{\tau}_2 + T_2 + \dots + T_{j-1}}} \\ &= \Gamma g c_{\tilde{u}_j} b_{\tilde{s}_j} a_{\tilde{\tau}_j}\end{aligned}$$

for

$$\begin{aligned}\check{u}_j &= u_j + \frac{\sigma_1 e^{-\tilde{\tau}_2 + T_2 + \dots + T_{j-1}}}{1 + \check{\rho}_j}, \\ \check{s}_j &= s_j + \eta_1 e^{-(\tilde{\tau}_2 + T_2 + \dots + T_{j-1})} + \check{\rho}_j (s_j + \eta_1 e^{-(\tilde{\tau}_2 + T_2 + \dots + T_{j-1})}), \\ \check{\tau}_j &= 2 \ln(1 + \check{\rho}_j);\end{aligned}$$

here $\check{\rho}_j = s_j \sigma_1 e^{-\tilde{\tau}_2 + T_2 + \dots + T_{j-1}} + \sigma_1 \eta_1$. Recall that $|\sigma_1| < \frac{6}{n} \varepsilon e^{\tilde{\tau}_2 - (T_2 + \dots + T_n)}$, $|\eta_1| < \frac{3}{n} \varepsilon$ and $|\tilde{\tau}_2| < \varepsilon^2$. Then $|\check{\rho}_j| < \frac{10}{n^2} \varepsilon^2 e^{-(T_j + \dots + T_n)}$ implies

$$|\check{\tau}_j| < \frac{1}{n^2} \varepsilon^2,\tag{3.56}$$

$$|\check{u}_j - u_j| < \frac{8}{n} \varepsilon e^{-(T_j + \dots + T_n)},\tag{3.57}$$

$$|\check{s}_j - s_j| < \frac{4}{n} \varepsilon e^{-(T_j + \dots + T_n)}. \quad (3.58)$$

Together with (3.53) and (3.54) we obtain

$$|\check{u}_1 - u_2|, |\check{u}_j - u_j| < \frac{28}{n^3} \varepsilon^3 + \frac{8}{n} \varepsilon e^{-(T_j + \dots + T_n)}, \quad j = 3, \dots, n, \quad (3.59a)$$

$$|\check{s}_j - s_j| < \frac{2}{n^3} \varepsilon^3 + \frac{4}{n} \varepsilon e^{-(T_j + \dots + T_n)}, \quad j = 1, 3, \dots, n, \quad (3.59b)$$

and as a consequence $|\check{s}_j| < \frac{1}{n-1} \varepsilon$ as well as $|\check{u}_j| < \frac{1}{n-1} \varepsilon$, for $j = 1, 3, \dots, n$. Therefore, $\check{y}_j := \varphi_{-\check{\tau}_j}(y_j) \in \mathcal{P}_{\frac{\varepsilon}{n-1}}(x)$ with $\check{y}_j = (\check{u}_j, \check{s}_j)_x$ for $j = 1, 3, \dots, n$. In addition, letting

$$\begin{aligned} \check{T}_2 &= -\check{\tau}_2 + T_2 + \check{\tau}_1 - \check{\tau}_3, \\ \check{T}_j &= T_j + \check{\tau}_j - \check{\tau}_{j+1} \quad \text{for } j = 3, \dots, n-1, \\ \check{T}_n &= \check{T} - (\check{T}_2 + \check{T}_3 + \dots + \check{T}_{n-1}), \end{aligned}$$

we have $\check{T} = \check{T}_2 + \dots + \check{T}_n$ and $\varphi_{\check{T}_2}(\check{y}_1) = \check{y}_3$, $\varphi_{\check{T}_j}(\check{y}_j) = \check{y}_{j+1}$ for $j = 3, \dots, n-1$ and $\varphi_{\check{T}_n}(\check{y}_n) = \check{y}_1$. This means that the orbit c_1 has an $(\frac{\varepsilon}{n-1}, n-1)$ -encounter.

Next we derive an estimate for $|\check{T}_j - T_j|$ that will be helpful. A short calculation shows that

$$T_2 - \check{T}_2 = \check{\tau}_2 - \check{\tau}_1 + \check{\tau}_3, \quad (3.60a)$$

$$T_j - \check{T}_j = \check{\tau}_{j+1} - \check{\tau}_j, \quad \text{for } j = 3, \dots, n-1, \quad (3.60b)$$

$$T_n - \check{T}_n = T - \check{T}_1 - \check{T}_1 + \check{\tau}_1 - \check{\tau}_n. \quad (3.60c)$$

By (3.41), (3.55), (3.56), and (3.50), we have

$$|\check{T}_2 - T_2| < \frac{25}{n^2} \varepsilon^2, \quad |\check{T}_j - T_j| < \frac{2}{n^2} \varepsilon^2, \quad j = 3, \dots, n-1, \quad \text{and} \quad |\check{T}_n - T_n| < \frac{19}{n^2} \varepsilon^2. \quad (3.61)$$

In summary,

$$|\check{T}_j - T_j| < \frac{25}{n^2} \varepsilon^2 \quad \text{for } j = 2, \dots, n. \quad (3.62)$$

Step 2: Applying the inductive assumption. The orbit c_1 is expressed by the bijection $\hat{P} : \{1, 3, \dots, n\} \rightarrow \{2, 3, \dots, n\}$ defined by $\hat{P} = \begin{pmatrix} 1 & 3 & \dots & n \\ 2 & 3 & \dots & n \end{pmatrix}$; see the solid line in Figure 8 (b). The orbit loops of c_1 are illustrated by $\hat{P}_{\text{loop}} = P_{1,\text{loop}} = \begin{pmatrix} 2 & 3 & \dots & n-1 & n \\ 3 & 4 & \dots & n & 1 \end{pmatrix}$. Let $\check{P} = P_1$ be defined by (3.34):

$$\check{P}(j) = \begin{cases} P(j) & \text{if } j \neq P^{-1}(1), \\ P(2) & \text{if } j = P^{-1}(1). \end{cases}$$

Then $\hat{P}_{\text{loop}}\check{P}$ is a permutation of the set $\{2, 3, \dots, n\}$ and is a single cycle since $P_{\text{loop}}P$ is a single cycle of the set $\{1, 2, \dots, n\}$. By the inductive assumption, the orbit c_1 has a partner orbit c'_1 which is illustrated by \check{P} ; see the solid line in Figure 8 (c). We see that the sequence $\{\check{P}_k\}_{k=0, \dots, n-4}$ generated

by \check{P} satisfies $\check{P}_k = P_{k+1}$ for $k = 0, \dots, n-4$; recall the definition of P_k in (3.34). Let \hat{T} be the period of c'_1 , then

$$\left| \frac{\hat{T} - \check{T}}{2} - \Delta \check{S}_{n-1} \right| < \omega_{n-1} \varepsilon^4 + \kappa_{n-1} \varepsilon^2 (e^{-\check{T}_2} + \dots + e^{-\check{T}_n}), \quad (3.63)$$

where

$$\begin{aligned} \Delta \check{S}_{n-1} &= \ln(1 + (\check{u}_3 - \check{u}_1)(\check{s}_3 - \check{s}_1)) + \sum_{j=3}^{n-2} \ln(1 + (\check{u}_{j+1} - \check{u}_j)(\check{s}_{j+1} - \check{s}_1)) \\ &\quad + \ln(1 + (\check{u}_{\check{P}(3)} - \check{u}_1)(\check{s}_{\check{P}^{-1}(2)} - \check{s}_3)) + \sum_{j=2}^{n-3} \ln(1 + (\check{u}_{\check{P}_{j-1}(j+2)} - \check{u}_{j+1})(\check{s}_{\check{P}_{j-1}^{-1}(j+1)} - \check{s}_{j+2})) \\ &= \ln(1 + (\check{u}_3 - \check{u}_1)(\check{s}_3 - \check{s}_1)) + \sum_{j=3}^{n-2} \ln(1 + (\check{u}_{j+1} - \check{u}_j)(\check{s}_{j+1} - \check{s}_1)) \\ &\quad + \ln(1 + (\check{u}_{P_1(3)} - \check{u}_1)(\check{s}_{P_1^{-1}(2)} - \check{s}_3)) + \sum_{j=3}^{n-2} \ln(1 + (\check{u}_{P_{j-1}(j+1)} - \check{u}_j)(\check{s}_{P_{j-1}^{-1}(j)} - \check{s}_{j+1})) \end{aligned}$$

and $\omega_{n-1}, \kappa_{n-1}$ are defined by (3.38). Denote

$$\Delta S_{n-1} = \sum_{j=2}^{n-2} \ln(1 + (u_{j+1} - u_j)(s_{j+1} - s_1)) + \sum_{j=2}^{n-2} \ln(1 + (u_{P_{j-1}(j+1)} - u_j)(s_{P_{j-1}^{-1}(j)} - s_{j+1})).$$

For $j, i, m, l \in \{1, 3, \dots, n\}$, using (3.59), we can show that

$$|(\check{u}_j - \check{u}_i)(\check{s}_l - \check{s}_m) - (u_j - u_i)(s_l - s_m)| < \frac{120}{(n-1)n^3} \varepsilon^4 + \frac{52}{n(n-1)} \varepsilon^2 e^{-T_n}. \quad (3.64)$$

Then

$$\begin{aligned} |\Delta \check{S}_{n-1} - \Delta S_{n-1}| &\leq 3 |(\check{u}_3 - \check{u}_1)(\check{s}_3 - \check{s}_1) - (u_3 - u_2)(s_3 - s_1)| \\ &\quad + 3 |(u_{P_1(3)} - u_2)(s_{P_1^{-1}(2)} - s_3) - (\check{u}_{P_1(3)} - \check{u}_1)(\check{s}_{P_1^{-1}(2)} - \check{s}_3)| \\ &\quad + 3 \sum_{j=3}^{n-2} |(u_{j+1} - u_j)(s_{j+1} - s_1) - (\check{u}_{j+1} - \check{u}_j)(\check{s}_{j+1} - \check{s}_1)| \\ &\quad + 3 \sum_{j=3}^{n-2} |(\check{u}_{P_{j-1}(j+1)} - \check{u}_j)(\check{s}_{P_{j-1}^{-1}(j)} - \check{s}_{j+1}) - (u_{P_{j-1}(j+1)} - u_j)(s_{P_{j-1}^{-1}(j)} - s_{j+1})| \\ &\leq 2(n-3) \cdot 3 \left(\frac{120}{(n-1)n^3} \varepsilon^4 + \frac{52}{n(n-1)} \varepsilon^2 e^{-T_n} \right) \leq \frac{720}{n^3} \varepsilon^4 + \frac{312}{n} \varepsilon^2 e^{-T_n} \end{aligned}$$

implies

$$\left| \frac{\hat{T} - \check{T}}{2} - \Delta S_{n-1} \right| \leq \left| \frac{\hat{T} - \check{T}}{2} - \Delta \check{S}_{n-1} \right| + |\Delta \check{S}_{n-1} - \Delta S_{n-1}|$$

$$\leq \left(\omega_{n-1} + \frac{720}{n^3}\right)\varepsilon^4 + \left(\kappa_{n-1} + \frac{312}{n}\right)\varepsilon^2(e^{-T_1} + \dots + e^{-T_n}) \quad (3.65)$$

due to (3.63).

Also by the inductive assumption, the partner orbit c'_1 has an $(n-1, \frac{3\varepsilon}{n-1})$ -encounter. More precisely, there are $z_1, z_3, \dots, z_n \in c'_1, z_j \in \mathcal{P}_{\frac{3\varepsilon}{n-1}}(x)$ for $z_j = (\hat{u}_j, \hat{s}_j)_x$ and $\hat{T}_2, \dots, \hat{T}_n > 0$ such that

$$|\hat{T}_j - \check{T}_j| \leq \Delta T_{(n-1)}\varepsilon^2, \quad (3.66)$$

$$\varphi_{\hat{T}_{P_1(j)}}(z_j) = \begin{cases} z_{P_1(j)+1} & \text{if } P_1(j) \neq n \\ z_1 & \text{if } P_1(j) = n, \end{cases} \quad (3.67)$$

$$d_X(\varphi_t(\check{y}_{P_1(j)}), \varphi_t(z_j)) < \Delta d_{(n-1)}\varepsilon \quad \text{for all } t \in [0, \max\{\check{T}_{P_1(j)}, \hat{T}_{P_1(j)}\}], \quad (3.68)$$

and

$$|\hat{u}_j - \check{u}_{P_1(j)}| < \alpha_{n-1}\varepsilon^3 + \beta_{n-1}\varepsilon(e^{-\check{T}_2} + \dots + e^{-\check{T}_n}), \quad (3.69a)$$

$$|\hat{s}_j - \check{s}_j| < \alpha_{n-1}\varepsilon^3 + \beta_{n-1}\varepsilon(e^{-\check{T}_2} + \dots + e^{-\check{T}_n}). \quad (3.69b)$$

with the convention

$$\check{y}_2 = \check{y}_1 \quad \text{as well as} \quad \check{u}_2 = \check{u}_1. \quad (3.70)$$

Recall that $\tilde{\tau}_2 = -2 \ln(1 - s_1(u_2 - u_1))$, $\tilde{\tau}_j = 2 \ln(1 + \check{\rho}_j)$. Furthermore, according to the proof of the Anosov closing lemma II, $T - \tilde{T}_1 - \check{T}_1 = 2 \ln(1 + (\tilde{s} + \eta_1)\tilde{u}_2)$. Whence, it follows from (3.60) that

$$\begin{aligned} e^{T_2 - \check{T}_2} &= (1 - s_1(u_2 - u_1))^{-2}(1 + \check{\rho}_1)^{-2}(1 + \check{\rho}_3)^2, \\ e^{T_j - \check{T}_j} &= (1 + \check{\rho}_{j+1})^2(1 + \check{\rho}_j)^{-2}, \quad j = 3, \dots, n-1, \\ e^{T_n - \check{T}_n} &= (1 + (\tilde{s}_2 + \eta_1)\tilde{u}_2)^2(1 + \check{\rho}_1)^2(1 + \check{\rho}_n)^{-2}. \end{aligned}$$

Then

$$e^{T_2 - \check{T}_2} \leq (1 + 4|s_1(u_2 - u_1)|)(1 + 4|\check{\rho}_3 - \check{\rho}_1|) < 1 + \frac{22}{n^2}\varepsilon^2.$$

Similarly, we obtain

$$e^{T_j - \check{T}_j} < 1 + \frac{1}{n^2}\varepsilon^2 \quad \text{for } j = 3, \dots, n-1 \quad \text{and} \quad e^{T_n - \check{T}_n} < 1 + \frac{18}{n^2}\varepsilon^2.$$

Therefore, it follows from (3.69) that

$$\begin{aligned} |\hat{u}_j - \check{u}_{P_1(j)}| &< \alpha_{n-1}\varepsilon^3 + \left(\beta_{n-1} + \frac{1}{n}\right)\varepsilon(e^{-T_2} + \dots + e^{-T_n}) \\ |\hat{s}_j - \check{s}_j| &< \alpha_{n-1}\varepsilon^3 + \left(\beta_{n-1} + \frac{1}{n}\right)\varepsilon(e^{-T_2} + \dots + e^{-T_n}); \end{aligned}$$

note that $\beta_{n-1}\frac{22}{n^2}\varepsilon^2 < \frac{1}{n}$. Together with (3.59), this shows that

$$|\hat{u}_j - u_{P_1(j)}| \leq \left(\alpha_{n-1} + \frac{28}{n^3}\right)\varepsilon^3 + \left(\beta_{n-1} + \frac{9}{n}\right)\varepsilon(e^{-T_2} + \dots + e^{-T_n}), \quad (3.71a)$$

$$|\hat{s}_j - s_j| \leq \left(\alpha_{n-1} + \frac{2}{n^3}\right)\varepsilon^3 + \left(\beta_{n-1} + \frac{5}{n}\right)\varepsilon(e^{-T_2} + \dots + e^{-T_n}). \quad (3.71b)$$

Step 3: Construction of the partner c' and proof of (3.37). We use the connecting lemma to ‘connect’ the orbit c'_1 and the orbit of $y_2 = \Gamma h_2$ to obtain the new partner orbit c' . To do this, we need to verify that the stretch $P^{-1}(1)$ -th contains a point (called $\bar{z}_{P^{-1}(1)}$) that lies on the Poincaré section of y_2 . Recall that $\Gamma g_1 = \Gamma g c_{u_1} b_{s_1}$ and $\Gamma h_2 = \Gamma g_1 c_{\sigma_2} b_{\eta_2}$. Applying Lemma 2.5, we can write

$$\begin{aligned} z_{P^{-1}(1)} &= \Gamma g c_{\bar{u}_{P^{-1}(1)}} b_{\bar{s}_{P^{-1}(1)}} = \Gamma g_1 c_{\sigma_2} b_{\eta_2} (b_{-\eta_2} c_{-\sigma_2} b_{-s_1} c_{-u_1 + \bar{u}_{P^{-1}(1)}} b_{\bar{s}_{P^{-1}(1)}}) \\ &= \Gamma h_2 c_{\bar{u}_{P^{-1}(1)}} b_{\bar{s}_{P^{-1}(1)}} a_{\bar{\tau}_{P^{-1}(1)}} \end{aligned}$$

and a short estimate shows that

$$|\bar{\tau}_{P^{-1}(1)}| < 4|\bar{\rho}_{P^{-1}(1)}| < \frac{44}{n^2} \varepsilon^2, \quad (3.72)$$

$$|\bar{u}_{P^{-1}(1)} - (-u_1 + \hat{u}_{P^{-1}(1)})| < \frac{72}{n^3} \varepsilon^3 + \frac{8}{n} \varepsilon e^{-T_1}, \quad (3.73)$$

$$|\bar{s}_{P^{-1}(1)} - (-\eta_2 - s_1 + \hat{s}_{P^{-1}(1)})| < \frac{100}{n^3} \varepsilon^3. \quad (3.74)$$

So that $|\bar{s}_{P^{-1}(1)}| < \frac{6\varepsilon}{n}$, $|\bar{u}_{P^{-1}(1)}| < \frac{5\varepsilon}{n}$ as well as

$$\bar{z}_{P^{-1}(1)} := \varphi_{-\bar{\tau}_{P^{-1}(1)}}(z_{P^{-1}(1)}) = \Gamma h_2 c_{u_{P^{-1}(1)}} b_{s_{P^{-1}(1)}} \in \mathcal{P}_{\frac{6\varepsilon}{n}}(y_2).$$

Now we can apply the connecting lemma (Lemma 2.3) to connect the $P^{-1}(1)$ -th entrance port of c'_1 and the 1st exit port of the orbit c_2 through y_2 ; see the red thick lines in Figure 8 (c) & (d). There are $T' \in \mathbb{R}$, $w_{P^{-1}(1)} = \Gamma h_2 c_{\bar{u}_{P^{-1}(1)} e^{-\hat{T}_1 + \sigma}} b_{\eta} \in \mathcal{P}_{\frac{12\varepsilon}{n}}(y_2)$ so that $\varphi_{T'}(w_{P^{-1}(1)}) = w_{P^{-1}(1)}$,

$$\begin{aligned} \left| \frac{T' - (\hat{T}_1 + \hat{T})}{2} - \ln(1 + \bar{u}_{P^{-1}(1)} \bar{s}_{P^{-1}(1)}) \right| &< \frac{90}{n^2} \varepsilon^2 (e^{-\hat{T}_1} + e^{-\hat{T}}) + \frac{240}{n^2} \varepsilon^2 e^{-\hat{T}_1 - \hat{T}} \\ &< \frac{92}{n^2} \varepsilon^2 (e^{-T_1} + e^{-T_2}), \end{aligned} \quad (3.75)$$

$$d_X(\varphi_t(w_{P^{-1}(1)}), \varphi_t(y_2)) < \frac{30}{n} \varepsilon \quad \text{for all } t \in [0, \hat{T}_1], \quad (3.76)$$

$$d_X(\varphi_{t+\hat{T}_1}(w_{P^{-1}(1)}), \varphi_t(\bar{z}_{P^{-1}(1)})) < \frac{30}{n} \varepsilon \quad \text{for all } t \in [0, \hat{T}], \quad (3.77)$$

and

$$|\eta - \bar{s}_{P^{-1}(1)}| < \frac{361}{n^3} \varepsilon^3, \quad |\sigma| < \frac{10}{n} \varepsilon e^{-\hat{T}_1 - \hat{T}}. \quad (3.78)$$

The partner orbit given by the permutation P is illustrated in Figure 8 (d). Now we prove (3.37). First, we have

$$\begin{aligned} &|\bar{s}_{P^{-1}(1)} - (s_{P^{-1}(1)} - s_2)| \\ &\leq |\bar{s}_{P^{-1}(1)} - (-\eta_2 - s_1 + \hat{s}_{P^{-1}(1)})| + |\hat{s}_{P^{-1}(1)} - s_{P^{-1}(1)}| + |\tilde{s}_2 - \eta_2| + |(s_2 - s_1) - \tilde{s}_2| \end{aligned}$$

$$\leq \left(\alpha_{n-1} + \frac{140}{n^3}\right)\varepsilon^3 + \left(\beta_{n-1} + \frac{5}{n}\right)\varepsilon(e^{-T_1} + \dots + e^{-T_n}) \quad (3.79)$$

due to (3.74), (3.71b), (3.45), and (3.41);

$$\begin{aligned} |\bar{u}_{P^{-1}(1)} - (u_{P(2)} - u_1)| &\leq |\bar{u}_{P^{-1}(1)} - (-u_1 + \hat{u}_{P^{-1}(1)})| + |\hat{u}_{P^{-1}(1)} - u_{P(2)}| \\ &\leq \left(\alpha_{n-1} + \frac{100}{n^3}\right)\varepsilon^3 + \left(\beta_{n-1} + \frac{9}{n}\right)\varepsilon(e^{-T_1} + \dots + e^{-T_n}) \end{aligned} \quad (3.80)$$

using (3.71a), (3.73) and noting that $P_1(P^{-1}(1)) = P(2)$. This yields

$$\begin{aligned} &|\ln(1 + \bar{u}_{P^{-1}(1)}\bar{s}_{P^{-1}(1)}) - \ln(1 + (u_{P(2)} - u_1)(s_{P^{-1}(1)} - s_2))| \\ &< \frac{21\alpha_n}{n}\varepsilon^4 + \frac{21\beta_n}{n}\varepsilon^2(e^{-T_1} + \dots + e^{-T_n}); \end{aligned}$$

note that $\alpha_{n-1} + \frac{140}{n^3} < \alpha_n$ and $\beta_{n-1} + \frac{9}{n} < \beta_n$. Together with (3.75), we obtain

$$\begin{aligned} &\left|\frac{T' - (\hat{T}_1 + \hat{T})}{2} - \ln(1 + (u_{P(2)} - u_1)(s_{P^{-1}(1)} - s_2))\right| \\ &\leq \frac{21\alpha_n}{n}\varepsilon^4 + \left(\frac{21\beta_n}{n} + \frac{92}{n^2}\right)\varepsilon^2(e^{-T_1} + \dots + e^{-T_n}). \end{aligned} \quad (3.81)$$

Now we are in a position to derive the action difference between c and c' . Note that

$$\Delta S_n = \Delta S_{n-1} + \ln(1 + (u_2 - u_1)(s_2 - s_1)) + \ln(1 + (u_{P(2)} - u_1)(s_{P^{-1}(1)} - s_2)).$$

By (3.47), (3.50), (3.65), and (3.81), the action difference between c and c' satisfies

$$\begin{aligned} \left|\frac{T' - T}{2} - \Delta S_n\right| &\leq \left|\frac{\hat{T}_1 - \tilde{T}_1}{2} - \ln(1 + (u_2 - u_1)(s_2 - s_1))\right| + \left|\frac{\tilde{T} - (T - \tilde{T}_1)}{2}\right| \\ &+ \left|\frac{\hat{T} - \tilde{T}}{2} - \Delta S_n\right| + \left|\frac{T' - (\hat{T}_1 + \hat{T})}{2} - \ln(1 + (u_{P(2)} - u_1)(s_{P^{-1}(1)} - s_2))\right| \\ &\leq \omega_n\varepsilon^4 + \kappa_n\varepsilon^2(e^{-T_1} + \dots + e^{-T_n}); \end{aligned}$$

note that by (3.38), we have

$$\omega_n = \omega_{n-1} + \frac{12}{n^4} + \frac{720}{n^3} + \frac{21\alpha_n}{n} \quad \text{and} \quad \kappa_n = \kappa_{n-1} + \frac{118}{n^2} + \frac{312}{n} + \frac{21\beta_n}{n}.$$

Step 4: Definition of v_j , T'_j and proof of (a). It has not been sufficient yet to prove that $|u'_j| < \frac{3}{n}\varepsilon$ and $|s'_j| < \frac{3}{n}\varepsilon$. We will do it after verifying (d). For simplicity, let us write $v_j \in \mathcal{P}_{\frac{3\varepsilon}{n}}(x)$ if we derive $v_j = \Gamma g c_{u'_j} b_{s'_j}$ for some $u'_j, s'_j \in \mathbb{R}$. In what follows we often use the fact that $|\sigma_1| < \frac{7}{n}\varepsilon e^{-(T_2 + \dots + T_n)}$, $|\eta_1| < \frac{3}{n}\varepsilon$, $|\eta_2| < \frac{3}{n}\varepsilon$, $|\sigma_2| < \frac{6}{n}\varepsilon e^{-\tilde{T}_1}$, $|\eta| < \frac{7}{n}\varepsilon$, $|s_1 + \eta_2| < \frac{1}{n}\varepsilon$, $|\bar{u}_{P^{-1}(1)}| < \frac{5}{n}\varepsilon$, $|\bar{s}_{P^{-1}(1)}| < \frac{6}{n}\varepsilon$.

Step 4.1: Definition of $v_{P^{-1}(1)}$. Recall $w_{P^{-1}(1)} = \Gamma h_2 c_{\bar{u}_{P^{-1}(1)} e^{-\hat{T}_1 + \sigma}} b_\eta$ and $\Gamma h_2 = \Gamma g_1 c_{\sigma_2} b_{\eta_2}$. Then applying Lemma 2.5, we can write

$$w_{P^{-1}(1)} = \Gamma g_1 c_{\sigma_2} b_{\eta_2} c_{\bar{u}_{P^{-1}(1)} e^{-\hat{T}_1 + \sigma}} b_\eta = \Gamma g(c_{u_1} b_{s_1} c_{\sigma_2} b_{\eta_2} c_{\bar{u}_{P^{-1}(1)} e^{-\hat{T}_1 + \sigma}} b_\eta)$$

$$= \Gamma g c_{u'_{P^{-1}(1)}} b_{s'_{P^{-1}(1)}} a_{\tau'_{P^{-1}(1)}}$$

and a short calculation shows

$$|\tau'_{P^{-1}(1)}| < \frac{1}{n^2} \varepsilon^2, \quad (3.82)$$

$$|u'_{P^{-1}(1)} - u_1| < \frac{15}{n} \varepsilon e^{-T_1}, \quad (3.83)$$

$$|s'_{P^{-1}(1)} - (s_1 + \eta_2 + \eta)| < \frac{5}{n^3} \varepsilon^3. \quad (3.84)$$

Together with (3.78), (3.74), and (3.71b), we obtain

$$\begin{aligned} |s'_{P^{-1}(1)} - s_{P^{-1}(1)}| &\leq |s'_{P^{-1}(1)} - (s_1 + \eta_2 + \eta)| + |\eta - \bar{s}_{P^{-1}(1)}| \\ &\quad + |\bar{s}_{P^{-1}(1)} - (-\eta_2 - s_1 + \hat{s}_{P^{-1}(1)})| + |\hat{s}_{P^{-1}(1)} - s_{P^{-1}(1)}| \\ &\leq \left(\alpha_{n-1} + \frac{468}{n^3} \right) \varepsilon^3 + \left(\beta_{n-1} + \frac{5}{n} \right) \varepsilon (e^{-T_2} + \dots + e^{-T_n}) \\ &< \alpha_n \varepsilon^3 + \beta_n \varepsilon (e^{-T_1} + \dots + e^{-T_n}). \end{aligned} \quad (3.85)$$

Define

$$v_{P^{-1}(1)} = \varphi_{-\tau'_{P^{-1}(1)}}(w_{P^{-1}(1)}). \quad (3.86)$$

Then $v_{P^{-1}(1)} \in \mathcal{P}_{\frac{3\varepsilon}{n}}(x)$ and $v_{P^{-1}(1)} = (u'_{P^{-1}(1)}, s'_{P^{-1}(1)})_x$ satisfy (3.40) for $j = P^{-1}(1)$, due to (3.83) and (3.85).

Step 4.2: Definition of v_2 . Denoting $w_2 = \varphi_{\hat{T}_1}(w_{P^{-1}(1)})$ and recalling that $y_2 = \Gamma h_2$ is \hat{T}_1 -periodic, we apply Lemma 2.5 to write

$$\begin{aligned} w_2 &= \varphi_{\hat{T}_1}(w_{P^{-1}(1)}) = \Gamma h_2 c_{\bar{u}_{P^{-1}(1)} + \sigma e^{\hat{T}_1}} b_{\eta e^{-\hat{T}_1}} = \Gamma g c_{u_1} b_{s_1} c_{\sigma_2} b_{\eta_2} c_{\bar{u}_{P^{-1}(1)} + \sigma e^{\hat{T}_1}} b_{\eta e^{-\hat{T}_1}} \\ &= \Gamma g c_{u'_2} b_{s'_2} a_{\tau'_2}. \end{aligned}$$

Analogously, we obtain

$$\begin{aligned} |\tau'_2| &< \frac{52}{n^2} \varepsilon^2, \\ |s'_2 - (s_1 + \eta_2)| &\leq \frac{39}{n^3} \varepsilon^3 + \frac{10}{n} \varepsilon e^{-T_1}. \end{aligned} \quad (3.87)$$

Therefore it follows from (3.45) and (3.41) that

$$|s'_2 - s_2| \leq |s'_2 - (s_1 + \eta_2)| + |\eta_2 - \tilde{s}_2| + |\tilde{s}_2 - (s_2 - s_1)| < \frac{77}{n^3} \varepsilon^3 + \frac{17}{n} \varepsilon e^{-T_1},$$

which shows that s'_2 satisfies (3.40b) for $j = 2$. Furthermore, $|u'_2 - (u_1 + \bar{u}_{P^{-1}(1)})| \leq \frac{117}{n^3} \varepsilon^3 + \frac{8}{n} \varepsilon e^{-T_1}$ yields

$$|u'_2 - u_{P(2)}| \leq |u'_2 - (u_1 + \bar{u}_{P^{-1}(1)})| + |\bar{u}_{P^{-1}(1)} - (u_{P(2)} - u_1)|$$

$$\begin{aligned}
&\leq \left(\alpha_{n-1} + \frac{217}{n^3}\right)\varepsilon^3 + \left(\beta_{n-1} + \frac{17}{n}\right)(e^{-T_1} + \dots + e^{-T_n}) \\
&\leq \alpha_n \varepsilon^3 + \beta_n \varepsilon (e^{-T_1} + \dots + e^{-T_n});
\end{aligned}$$

recall (3.80). Hence u'_2 satisfies (3.40a) for $j = 2$. Defining

$$v_2 := \varphi_{-\tau'_2}(w_2) = \Gamma g c_{u'_2} b_{s'_2},$$

we have shown that $v_2 \in \mathcal{P}_{\frac{3\varepsilon}{n}}(x)$ and $v_2 = (u'_2, s'_2)$ satisfies (3.40). Step 4.3: Definition of v_j with $j \neq P^{-1}(1), j \neq 2$. Since $Q = P_{1,\text{loop}}P_1$ is a single-cycle-permutation of the set $\{1, 3, \dots, n\}$, we may write

$$Q = P_{1,\text{loop}}P_1 = (a, Q(a), Q^2(a), \dots, Q^{n-2}(a)),$$

where $a = P^{-1}(1)$. For every $j \in \{1, \dots, n\} \setminus \{P^{-1}(1), 2\}$, there is $k \in \{1, \dots, n-2\}$ such that $j = Q^k(a)$. Write $z_j = \Gamma \hat{h}_j$ for $\hat{h}_j \in \mathbf{G}$ and put

$$\hat{T}_k^a := \hat{T}_{P_1 Q^0(a)} + \hat{T}_{P_1 Q(a)} + \dots + \hat{T}_{P_1 Q^{k-1}(a)}. \quad (3.88)$$

Note that since (3.67), we have $z_j = z_{Q^k(a)} = \varphi_{\hat{T}_k^a}(z_{P^{-1}(1)})$. Set $w_j := \varphi_{\hat{T}_1 + \hat{T}_k^a}(w_{P^{-1}(1)})$. Then using $\Gamma \hat{h}_j = \Gamma g c_{\hat{u}_j} b_{\hat{s}_j}$ and Lemma 2.5, we write

$$\begin{aligned}
w_j &= \Gamma \hat{h}_{P^{-1}(1)} a_{\hat{T}_k^a} b_{-\bar{s}_{P^{-1}(1)} e^{-\hat{T}_k^a} c_{\sigma e^{\hat{T}_k^a}} b_{\eta e^{-\hat{T}_k^a}}} = \Gamma \hat{h}_j b_{-\bar{s}_{P^{-1}(1)} e^{-\hat{T}_k^a} c_{\sigma e^{\hat{T}_k^a}} b_{\eta e^{-\hat{T}_k^a}}} \\
&= \Gamma g c_{\hat{u}_j} b_{\hat{s}_j - \bar{s}_{P^{-1}(1)} e^{-\hat{T}_k^a} c_{\sigma e^{\hat{T}_k^a}} b_{\eta e^{-\hat{T}_k^a}}} = \Gamma g c_{u'_j} b_{s'_j a_{\tau'_j}}
\end{aligned}$$

and we have

$$|\tau'_j| < \frac{1}{n^2} \varepsilon^2, \quad |u'_j - \hat{u}_j| < \frac{30}{n} \varepsilon e^{-T_1}, \quad |s'_j - \hat{s}_j| < \frac{4}{n^3} \varepsilon^3. \quad (3.89)$$

Therefore, using (3.71), we obtain

$$\begin{aligned}
|u'_j - u_{P(j)}| &\leq |u'_j - \hat{u}_j| + |\hat{u}_j - u_{P(j)}| \\
&\leq \left(\alpha_{n-1} + \frac{30}{n^3}\right)\varepsilon^3 + \frac{30}{n} \varepsilon e^{-T_1} + \left(\beta_{n-1} + \frac{9}{n}\right)\varepsilon(e^{-T_2} + \dots + e^{-T_n}) \\
&\leq \alpha_n \varepsilon^3 + \beta_n \varepsilon (e^{-T_1} + \dots + e^{-T_n}) \\
|s'_j - s_j| &\leq |s'_j - \hat{s}_j| + |\hat{s}_j - s_j| \\
&= \left(\alpha_{n-1} + \frac{6}{n^3}\right)\varepsilon^3 + \left(\beta_{n-1} + \frac{5}{n}\right)\varepsilon(e^{-T_2} + \dots + e^{-T_n}) \\
&\leq \alpha_n \varepsilon^3 + \beta_n \varepsilon (e^{-T_1} + \dots + e^{-T_n}).
\end{aligned}$$

Define

$$v_j = \varphi_{-\tau'_j}(w_j) = \Gamma g c_{u'_j} b_{s'_j}$$

to obtain $v_j \in \mathcal{P}_{\frac{3\varepsilon}{n}}(x)$; and (3.90) and (3.90) show that $v_j = (u'_j, s'_j)_x$ satisfies (3.40).

Step 4.4: Definition of T'_1, \dots, T'_n and proof of (a). The definition of w_1, \dots, w_n together with (3.67) lead to

$$\varphi_{\hat{T}_j}(w_j) = \begin{cases} w_{P(j)+1} & \text{if } P(j) \neq n \\ w_1 & \text{if } P(j) = n \end{cases} \quad (3.90)$$

and $T' = \hat{T}_1 + \dots + \hat{T}_n$. Recall $v_j = \varphi_{-\tau'_j}(w_j)$, and define

$$T'_j = \begin{cases} \tau'_{P^{-1}(j)} + \hat{T}_j - \tau'_{j+1} & \text{if } j \neq n \\ \tau'_{P^{-1}(j)} + \hat{T}_j - \tau'_1 & \text{if } j = n. \end{cases} \quad (3.91)$$

Then $T' = T'_1 + \dots + T'_n$ and

$$\varphi_{T'_{P(j)}}(v_j) = \begin{cases} v_{P(j)+1} & \text{if } P(j) \neq n \\ v_1 & \text{if } P(j) = n. \end{cases} \quad (3.92)$$

Step 5: Proof of (c). Recall from (3.82), (3.87), and (3.89) that

$$|\tau'_{P^{-1}(1)}| < \frac{1}{n^2} \varepsilon^2, \quad |\tau'_2| < \frac{52}{n^2} \varepsilon^2, \quad |\tau'_j| < \frac{1}{n^2} \varepsilon^2 \quad \text{for } j \neq 2, j \neq P^{-1}(1). \quad (3.93)$$

Case 1: $j = 1$. We have

$$|T'_1 - T_1| \leq |T'_1 - \hat{T}_1| + |\hat{T}_1 - T_1| \leq |\tau'_{P^{-1}(1)}| + |\tau'_2| + |\hat{T}_1 - T_1| \leq \frac{82}{n^2} \varepsilon^2 < \Delta T_{(n)}$$

due to (3.93) and (3.46). Case 2: $j \neq 1$. The observation

$$|T'_j - \hat{T}_j| < \frac{53}{n^2} \varepsilon^2 \quad \text{for all } j = 1, \dots, n$$

yields

$$|T'_j - T_j| \leq |T'_j - \hat{T}_j| + |\hat{T}_j - \check{T}_j| + |\check{T}_j - T_j| \leq \Delta T_{(n-1)} + \frac{78}{n^2} \varepsilon^2 = \Delta T_{(n)}$$

by (3.66) and (3.62).

Step 6: Proof of (d). Note that $|T'_j - \hat{T}_j|, |T'_j - T_j| < \Delta T_{(n)} \varepsilon^2 < 78 \varepsilon^2$. Due to Lemma 2.7, it remains to show that

$$d_X(\varphi_t(v_j), \varphi_t(x_{P(j)})) < \frac{40}{n} \varepsilon + \Delta d_{(n-1)} \varepsilon \quad \text{for all } t \in [0, \min\{T_{P(j)}, \hat{T}_{P(j)}\}].$$

Here the argument is similar to Step 6 in the proof of the case $L = 3$. We first review some results that will be helpful. Firstly, recall that $\tilde{x}_2 = \varphi_{\tilde{\tau}_2}(x_2)$ and $y_j = \varphi_{-\tilde{\tau}_2+T_2+\dots+T_{j-1}}(y_1)$, $j = 3, \dots, n$. Then it follows from (3.52) and $|\tilde{\tau}_2| < \frac{8}{n^2} \varepsilon^2$ that

$$d_X(\varphi_t(y_1), \varphi_t(x_2)) < \frac{9}{n} \varepsilon + \frac{8}{n^2} \varepsilon^2 \quad \text{for all } t \in [0, T_2] \quad (3.94)$$

and

$$d_X(\varphi_t(y_j), \varphi_t(x_j)) < \frac{9}{n} \varepsilon \quad \text{for all } t \in [0, T_j + \dots + T_n], \quad (3.95)$$

for $j = 3, \dots, n$. Secondly, due to $w_2 = \varphi_{\hat{T}_1}(w_{P^{-1}(1)})$ and $w_j = \varphi_{\hat{T}_k^a}(w_2)$ for $j \neq P^{-1}(1), j \neq 2$, where \hat{T}_k^a is defined by (3.88), we have

$$\begin{aligned} d_X(\varphi_t(w_j), \varphi_t(z_j)) &= d_X(\varphi_{t+\hat{T}_k^a}(w_2), \varphi_{t+\hat{T}_k^a}(z_{P^{-1}(1)})) \\ &\leq d_X(\varphi_{t+\hat{T}_k^a}(w_2), \varphi_{t+\hat{T}_k^a}(\bar{z}_{P^{-1}(1)})) + d_X(\varphi_{t+\hat{T}_k^a}(\bar{z}_{P^{-1}(1)}), \varphi_{t+\hat{T}_k^a}(z_{P^{-1}(1)})) \\ &\leq \frac{30}{n} \varepsilon + |\bar{\tau}_{P^{-1}(1)}| < \frac{31}{n} \varepsilon \quad \text{for all } t \in [0, \hat{T}_{P(j)}], \end{aligned}$$

using (3.67) and (3.77); recall $\bar{z}_{P^{-1}(1)} = \varphi_{-\bar{\tau}_{P^{-1}(1)}}(z_{P^{-1}(1)})$.

Step 6.1: $j = P^{-1}(1)$. For $t \in [0, \min\{\hat{T}_1, \tilde{T}_1\}]$,

$$\begin{aligned} d_X(\varphi_t(v_{P^{-1}(1)}), \varphi_t(x_1)) &\leq d_X(\varphi_t(v_{P^{-1}(1)}), \varphi_t(w_{P^{-1}(1)})) + d_X(\varphi_t(w_{P^{-1}(1)}), \varphi_t(y_2)) + d_X(\varphi_t(y_2), \varphi_t(x_1)) \\ &< |\tau'_{P^{-1}(1)}| + \frac{30}{n} \varepsilon + \frac{9}{n} \varepsilon < \frac{39}{n} \varepsilon + \frac{1}{n^2} \varepsilon^2, \end{aligned}$$

due to (3.82), (3.76), and (3.49). Since $|\hat{T}_1 - \tilde{T}_1| < \frac{21}{n^2} \varepsilon^2$, we obtain

$$d_X(\varphi_t(v_{P^{-1}(1)}), \varphi_t(x_1)) < \frac{39}{n} \varepsilon + \frac{1}{n^2} \varepsilon^2 + \sqrt{2} \cdot \frac{21}{n^2} \varepsilon^2 < \frac{40}{n} \varepsilon \quad \text{for all } t \in [0, \hat{T}_1].$$

Step 6.2: $j = 2$. Recall that $|\tau'_2| < \frac{52}{n^2} \varepsilon^2$, $|\bar{\tau}_{P^{-1}(1)}| < \frac{44}{n^2} \varepsilon^2$, $|\check{\tau}_{P(2)}| < \frac{1}{n^2} \varepsilon^2$. Case 1: $P(2) \neq 2$. For $t \in [0, \min\{T_{P(2)}, \hat{T}_{P(2)}\}]$, we have

$$\begin{aligned} d_X(\varphi_t(v_2), \varphi_t(x_{P(2)})) &\leq d_X(\varphi_t(v_2), \varphi_t(w_2)) + d_X(\varphi_t(w_2), \varphi_t(\bar{z}_{P^{-1}(1)})) + d_X(\varphi_t(\bar{z}_{P^{-1}(1)}), \varphi_t(z_{P^{-1}(1)})) \\ &\quad + d_X(\varphi_t(z_{P^{-1}(1)}), \varphi_t(\check{y}_{P(2)})) + d_X(\varphi_t(\check{y}_{P(2)}), \varphi_t(y_{P(2)})) + d_X(\varphi_t(y_{P(2)}), \varphi_t(x_{P(2)})) \\ &\leq |\tau'_2| + |\bar{\tau}_{P^{-1}(1)}| + |\check{\tau}_{P(2)}| + d_X(\varphi_{t+\hat{T}_1}(w_{P^{-1}(1)}), \varphi_t(\bar{z}_{P^{-1}(1)})) \\ &\quad + d_X(\varphi_t(z_{P^{-1}(1)}), \varphi_t(\check{y}_{P(2)})) + d_X(\varphi_t(y_{P(2)}), \varphi_t(x_{P(2)})) \\ &< \frac{97}{n^2} \varepsilon^2 + \frac{30}{n} \varepsilon + \Delta d_{(n-1)} \varepsilon + \frac{9}{n} \varepsilon < \frac{40}{n} \varepsilon + \Delta d_{(n-1)} \varepsilon, \end{aligned}$$

using (3.77), (3.68), and (3.95). Case 2: $P(2) = 2$. Analogously to Case 1, by replacing $y_{P(2)}, \check{y}_{P(2)}$, and $\check{\tau}_{P(2)}$ by y_1, \check{y}_1 , and $\check{\tau}_1$, respectively, and using (3.94), we obtain

$$d_X(\varphi_t(v_2), \varphi_t(x_{P(2)})) < \frac{113}{n^2} \varepsilon^2 + \frac{39}{n} \varepsilon + \Delta d_{(n-1)} \varepsilon < \frac{40}{n} \varepsilon + \Delta d_{(n-1)} \varepsilon, \quad (3.96)$$

recalling $|\check{\tau}_1| < \frac{16}{n^2} \varepsilon^2$ from (3.55). Step 6.3: $j \neq P^{-1}(1), j \neq 2$. Case 1: $P(j) \neq 2$. For $t \in [0, \min\{T_{P(j)}, \hat{T}_{P(j)}\}]$, using (3.96), (3.68), and (3.95), we obtain

$$d_X(\varphi_t(v_j), \varphi_t(x_{P(j)})) \leq d_X(\varphi_t(v_j), \varphi_t(w_j)) + d_X(\varphi_t(w_j), \varphi_t(z_j)) + d_X(\varphi_t(z_j), \varphi_t(\check{y}_{P(j)}))$$

$$\begin{aligned}
& + d_X(\varphi_t(\check{y}_{P_1(j)}), \varphi_t(y_{P_1(j)})) + d_X(\varphi_t(y_{P_1(j)}), \varphi_t(x_{P_1(j)})) \\
& < |\tau'_j| + |\check{\tau}_{P(j)}| + \frac{31}{n} \varepsilon + \Delta d_{(n-1)} \varepsilon + \frac{9}{n} \varepsilon \\
& < \frac{40}{n} \varepsilon + \Delta d_{(n-1)} \varepsilon;
\end{aligned}$$

note that $|\tau'_j| < \frac{52}{n^2} \varepsilon^2$ by (3.93) and $|\check{\tau}_j| < \frac{16}{n^2} \varepsilon^2$ by (3.55) and (3.56). Case 2: $P(j) = 2$. Here the argument is similar to Case 2 in Step 6.2.

Step 7: We prove that $|u'_j| < \frac{3}{n} \varepsilon, |s'_j| < \frac{3}{n} \varepsilon$. By Step 5, it has been shown that

$$d_X(\varphi_t(v_j), \varphi_t(x_{P(j)})) < \Delta d_{(n)} \varepsilon < \varepsilon_* \quad \text{for all } t \in [0, \max\{T_{P(j)}, T'_{P(j)}\}] \quad (3.97)$$

for $j = 1, \dots, n$; where $\varepsilon_0 = \varepsilon(\rho)$ is from Theorem 2.1 with respect to $\rho = 1$ and ε_* is from Lemma 2.9. Then $\varepsilon_* < \varepsilon_0$ implies that $|u'_j - u_{P(j)}| < e^{-T_{P(j)}} < \frac{2}{n} \varepsilon$ as well as $|u'_j| < \frac{3}{n} \varepsilon$. Furthermore, recall (3.92) and $\varphi_{T_1}(x_1) = x_2, \varphi_{T_2}(x_2) = x_3, \dots, \varphi_{T_n}(x_n) = x_1$. It follows from (3.97) that

$$d_X(\varphi_t(v_{P(j)+1}), \varphi_t(x_{P(j)+1})) < \varepsilon_0 \quad \text{for all } t \in [-\max\{T_{P(j)}, T'_{P(j)}\}, 0]$$

for $j = 1, \dots, n$ with the convention $P(j) + 1 \equiv 1$ if $P(j) = n$. This means that

$$d_X(\varphi_t(v_j), \varphi_t(x_j)) < \varepsilon_0 \quad \text{for all } t \in [-\max\{T_{j-1}, T'_{j-1}\}, 0].$$

Then by Theorem 2.1, $|s'_j - s_j| < |s'_j s_j| |u'_j - u_j| + e^{-T_{j-1}}$ leads to

$$|s'_j| < \frac{|s_j| + e^{-T_{j-1}}}{1 - |s_j| |u'_j - u_j|} < \frac{\frac{\varepsilon}{n} + \frac{5\varepsilon}{3n}}{1 - \frac{\varepsilon}{n} \cdot \frac{4\varepsilon}{n}} < \frac{9}{8} \cdot \frac{8}{3n} \varepsilon = \frac{3}{n} \varepsilon.$$

Step 8: The distinction between c and c' . Recall from Step 7 that

$$|u'_j - u_{P(j)}| < e^{-T_{P(j)}} \quad (3.98)$$

and

$$|s'_j - s_j| < |s'_j s_j| |u'_j - u_j| + e^{-T_{j-1}} < \frac{3}{L^2} \varepsilon^2 \cdot \frac{4}{L} \varepsilon + e^{-T_{j-1}} = \frac{12}{L^3} \varepsilon^3 + e^{-T_{j-1}} \quad (3.99)$$

for $j = 1, \dots, L$. By assumption and what we have shown, c and c' cross the Poincaré section of x at $x_j = (u_j, s_j)_x$ and $v_j = (u'_j, s'_j)_x$, $j = 1, \dots, L$, respectively. If c and c' do coincide, then they will have the same intersections with the Poincaré section at x . For any $i, j \in \mathbb{N}_L := \{1, \dots, L\}$ given, we prove that $v_j \neq x_i$. Case 1: $P(j) \neq i$. Then by condition (ii) and (3.98) it follows that

$$\begin{aligned}
|u'_j - u_i| & \geq |u_i - u_{P(j)}| - |u_{P(j)} - u'_j| > \frac{6}{5} (e^{-T_{P(j)}} + e^{-T_i}) - e^{-T_{P(j)}} \\
& = \frac{1}{5} e^{-T_{P(j)}} + \frac{6}{5} e^{-T_i} > 0
\end{aligned}$$

and hence $u'_j \neq u_i$ as well as $v_j \neq x_i$. Case 2: $P(j) = i \neq j$. We prove that $s'_j \neq s_i$. Indeed, due to condition (ii) and (3.99), we have

$$\begin{aligned} |s'_j - s_i| &\geq |s_j - s_{P(j)}| - |s_j - s'_j| > \frac{24}{L^3} \varepsilon^3 + e^{-T_{j-1}} + e^{-T_{P(j)-1}} - \left(\frac{12}{L^3} \varepsilon^3 + e^{-T_{j-1}} \right) \\ &= \frac{12}{L^3} \varepsilon^3 + e^{-T_{P(j)-1}} > 0 \end{aligned}$$

so that $s'_j \neq s_i$ as well as $v_j \neq x_i$. Case 3: $P(j) = i = j$. For a contradiction, we suppose that $v_j = x_j$. Then the orbits c and c' do agree and they have the same intersections with $\mathcal{P}_\varepsilon(x)$. Denote by F_P the set of fixed points of P . For any $k \in F_P$, we show that $v_k = x_k$. Indeed, suppose that $v_{k_0} \neq x_{k_0}$ for some $k_0 \in F_P$. This implies that $v_{k_0} = x_{i_0}$ for some $i_0 \neq k_0$. Then $P(k_0) = k_0 \neq i_0$ which contradicts Case 1. Therefore $v_k = x_k$ for all $k \in F_P$. Since P is not the identity permutation, there is $k_0 \in \mathbb{N}_L$ so that $P(k_0) \neq k_0$. By above, $v_{k_0} = x_{l_0}$ for some $l_0 \in \mathbb{N}_L$, $P(l_0) \neq l_0$. If $P(k_0) \neq l_0$ we have a contradiction to Case 1. If $P(k_0) = l_0$ then we have a contradiction to Case 2. Therefore $v_j \neq x_j$ as was to be shown. \square

Remark 3.2. In order to show that c has $(L-1)! - 1$ partners, we have to make sure the partner orbits are pairwise distinct. Let Q be a permutation in S_L such that $P_{\text{loop}}Q$ is a single cycle and $Q \neq P$. By Theorem 3.2, the orbit c has a partner orbit $c'' \neq c$ which is described by the permutation Q . Furthermore, there are L points w_1, \dots, w_L in c'' such that $w_j \in \mathcal{P}_\varepsilon(x)$ with $w_j = (u''_j, s''_j)_x$ satisfying

$$|u''_j - u_{Q(j)}| < e^{-T_{Q(j)}}, \quad |s''_j - s_j| < \frac{12}{L^3} \varepsilon^3 + e^{-T_{j-1}} \quad (3.100)$$

for all $j = 1, \dots, L$ with the convention $T_{j-1} = T_L$ for $j = 1$. Then c'' is different from c . Now we prove that c' and c'' do not coincide. Fixing $i, j \in \mathbb{N}_L$, we show that $w_j \neq v_i$. Case 1: $Q(j) \neq P(i)$. Then due to condition (ii), (3.98), and (3.100) we have

$$\begin{aligned} |u''_j - u'_i| &\geq |u_{Q(j)} - u_{P(i)}| - |u''_j - u_{Q(j)}| - |u'_i - u_{P(i)}| \\ &> \frac{6}{5} (e^{-T_{Q(j)}} + e^{-T_{P(i)}}) - e^{-T_{Q(j)}} - e^{-T_{P(i)}} > 0; \end{aligned}$$

hence $u''_j \neq u'_i$ and as a consequence $w_j \neq v_i$. Case 2: $Q(j) = P(i)$ and $j \neq i$. Then

$$\begin{aligned} |s''_j - s'_i| &\geq |s_j - s_i| - |s''_j - s_j| - |s'_i - s_i| \\ &> \frac{24}{L^3} \varepsilon^3 + e^{-T_{j-1}} + e^{-T_{i-1}} - \left(\frac{12}{L^3} \varepsilon^3 + e^{-T_{j-1}} \right) - \left(\frac{12}{L^3} \varepsilon^3 + e^{-T_{i-1}} \right) = 0 \end{aligned}$$

implies that $s''_j \neq s_i$ and thus $w_j \neq v_i$. Case 3: $Q(j) = P(i)$ and $j = i$. For a contradiction, suppose that $w_j = v_i$. Then the orbits c' and c'' do agree, so that they have the same intersections with $\mathcal{P}_\varepsilon(x)$. Denote by I the subset of \mathbb{N}_L consisting of k so that $Q(k) = P(k)$. Note that I may be empty but $I \neq \mathbb{N}_L$ since $Q \neq P$. If $I \neq \emptyset$ then $w_k = v_k$ for all $k \in I$ since otherwise $w_{k_0} = v_{l_0}$ for some $k_0 \in I$ and $l_0 \notin I$ implies $Q(l_0) \neq P(k_0)$ which contradicts Case 1. Fix $k, k' \in \mathbb{N}_L \setminus I$ such that $w_k = v_{k'}$. According to Case 1, we have $Q(k) = P(k')$. This implies that $k \neq k'$ since $k, k' \notin I$. Then a contradiction to Case 2 is obtained, and therefore $w_j \neq v_i$. \diamond

Remark 3.3. (a) If $P(j) = j$ for some $j \in \{1, \dots, L\}$ then we count the L -encounter as an $(L - 1)$ -encounter, and the error in the estimate of action difference would be better.

(b) For the case $L = 3$, since the given orbit has a unique partner orbit, condition (ii) in Theorem 3.1 can be reduced to $|u_1 - u_2| > e^{-T_2}$, $|u_1 - u_3| > e^{-T_3}$, and $|s_1 - s_3| > \frac{4}{9}\varepsilon^3 + e^{-T_2}$. Due to Theorem 2.1, Condition (ii) in Theorem 3.2 makes sure that any two encounter stretches are not too close for the whole time before intersecting the Poincaré section again, i.e., they do not simultaneously approach a shorter periodic orbit. In other words, the stretches are separated by non-vanishing loops (also called intervening loops or the stretches do not overlap), also needed in physics literatures, see [15, 18].

(c) If a periodic orbit has several encounters, for example one L -encounter and one N -encounter, $L, N \geq 3$, the partner orbits can be constructed as the following. We apply Theorem 3.2 for the L -encounter to have a new periodic orbit. This orbit has a N -encounter whose entrance ports and exit ports are like the ones of the original orbit. Then we can apply Theorem 3.2 for this encounter and we will obtain the corresponding partner for the original one. \diamond

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